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Why Does Data Reject the *Lucas Critique*?

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ABSTRACT. – This paper examines the consequences for macroeconomic theory of the *Favero-Hendry* finding that the *Lucas critique* of econometric policy evaluation is rejected by the data. I revisit the idea that this failure may be explained by models with indeterminate equilibria and I develop a class of expectations rules that I call generalized adaptive expectations. I illustrate how these rules may be implemented in a series of examples.

Pourquoi les données rejettent-elles la critique de Lucas ?

RÉSUMÉ. – Cet article étudie les conséquences, pour la théorie macroéconomique, du rejet par les données, établi par *Favero* et *Hendry*, de la critique émise par *Lucas* à propos de l'évaluation économétrique des politiques économiques. Je reviens sur l'idée que cette mise en défaut peut être expliquée par des modèles à équilibres indéterminés et je développe une classe de règles d'anticipations que j'appelle anticipations adaptatives généralisées. J'illustre comment ces règles peuvent être mises en application dans une série d'exemples.

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1 Introduction

In the 1970's, Robert E. LUCAS Jr. suggested that the parameter estimates of econometric models were unstable because of rational expectations. This argument, now known as the *Lucas Critique*, leads to strong predictions. Specifically, if expectations are rational, and if the economy is well described by a dynamic general equilibrium model with a locally unique equilibrium, one would expect to find 'cross equation restrictions' linking the parameters of econometric models.

Using an indirect test of the *Lucas Critique*, FAVERO and HENDRY [1992] have claimed that it fails in practice. Their claim, based on the idea of super-exogeneity, isolates examples of regime changes. In rational expectations models, the cross equation restrictions imply that following a structural break in the money supply rule, one should find corresponding structural breaks in the equations describing the endogenous variables of the model. In UK data *Favero* and *Hendry* find that when the money supply process changes there is no such accompanying change in the process describing money demand. They infer that rational expectations is at fault. In a comprehensive survey of related literature, ERICSSON and IRONS [1995] report that many of the papers that have used super-exogeneity tests to study the *Lucas Critique* have found similar results.

In a response to the FAVERO-HENDRY paper [1992], I suggested that the failure of super-exogeneity that they report may not be a failure of rational expectations, but instead a failure of a specific class of rational expectations models that impose determinacy of the equilibrium. In this paper, I develop my previous argument in three directions. Firstly, I compare the indeterminacy approach to a popular alternative based on the idea that expectations are formed by adaptive learning. Secondly, I answer a criticism of the indeterminacy approach by proposing a theory of how expectations are formed in practice. I call this approach, generalized adaptive expectations. Thirdly, I show how to apply generalized adaptive expectations in a series of examples.

2 Failure of the *Lucas Critique*

In this section, I provide an example of an economic model and use it to illustrate the arguments of *Favero* and *Hendry*. I will take it as given for the purpose of this paper that these results are correct and that the *Lucas Critique* really does fail in practice although this assertion is by no means uncontentious.[1] My example has the following components. The world has an economic and a political structure. The economic structure is stable and is

1. Several authors have focused on the weakness of the super-exogeneity test in small samples. See, for example, LINDE [2000] and COLLARD *et al.* [2001].

captured by a rational expectations market clearing model. The political structure is stable over the 'medium term' but is subject to occasional unpredictable shifts. These are called 'regime changes'. After the occurrence of a regime change private agents recalculate the rules that they use to form their expectations of future variables; during this period of recalculation, expectations may not be rational.

2.1 An Example

To fix ideas I will adapt a model drawn from Phillip CAGAN's work [1956] on hyperinflations.

$$p_t = \alpha E_t\left[p_{t+1}\right] + \beta m_t + e_t^1, \tag{1}$$

$$m_t = \mu m_{t-1} + e_t^2, \tag{2}$$

$$\lim_{t\to\infty} |m_t - p_t| < \infty. \tag{3}$$

In this example p_t, is the log of the price level, m_t, is the log of the money supply, α and β are structural parameters derived from some underlying economic model of behavior, μ is a policy parameter and e_t^1 and e_t^2 are random shocks with zero conditional means. Equation (1) is a reduced form that represents the economic structure and Equation (2) is a rule that represents the policy followed by the monetary authority. The term $E_t\left[p_{t+1}\right]$ represents the expectation of the future price and the rational expectations assumption implies that agents use the actual probability distribution of future realizations of the log price to calculate this expectation. The inequality (3) is a boundedness condition that typically follows from the transversality condition of an individual optimizing model.

Under the assumption that $\alpha\mu < 1$, the rational expectations solution to this model is found by iterating equation (1) forwards and substituting for future values of m_t using the policy rule (2). This leads to the rational expectations solution,

$$p_t = \frac{\beta m_t}{1-\alpha\mu}. \tag{4}$$

Suppose that an econometrician were to construct a VAR using the two variables p_t and m_t by estimating a system of equations of the form:

$$\begin{aligned} p_t &= a_{11} p_{t-1} + a_{12} m_{t-1} + u_t, \\ m_t &= a_{21} p_{t-1} + a_{22} m_{t-1} + v_t. \end{aligned} \tag{5}$$

If the rational expectations model is correct, since the behavior of money and the price level should be governed by equations (2) and (4), the econometrician should expect to observe the parameter restrictions,

$$a_{11} = 0, a_{12} = \frac{\mu\beta}{1 - \alpha\mu}, \tag{6}$$
$$a_{21} = 0, a_{22} = \mu.$$

The important feature of these restrictions is the appearance of μ in the term a_{12}. This implies that the behavior of the price level depends on the policy rule and it is this dependence of reduced form parameters on policy parameters that HANSEN and SARGENT [1980] have called a 'hallmark' of rational expectations models.

In practice, one can test rational expectations models by estimating VARs and checking if the restrictions implied by theory hold in practice. There are many ways of implementing such tests most of which lead to resounding rejections of the theory. In the following section, I will briefly summarize a test of rational expectations models carried out by *Carlo Favero* and *David Hendry* and I will evaluate two alternative ways of explaining the results that they report.

2.2 The Evidence of *Favero* and *Hendry*

The *Favero* and *Hendry* critique exploits the cross equations restriction of a rational expectations model to develop an empirical test of the theory. *Favero* and *Hendry* argue that, if there is a structural break in the parameters of the policy rule, there must also be a structural break in the parameters of the behavioral equation. In estimates of UK money demand they find evidence of a structural break in the money supply rule, but are able to fit a constant parameter money demand function for the entire sample period without evidence of any such break. *Favero* and *Hendry* conclude, (I will argue incorrectly), that the assumption of rational expectations cannot hold in these circumstances.[2]

In my example, the regime change discovered by *Favero* and *Hendry* would be represented by a change in μ. Suppose that for some period of time the policy parameter is equal to μ_A and that at some unforeseen date, T, it changes to μ_B. An econometrician estimating the parameters of this economy would discover that estimates of the parameter μ, from regressing m_t on m_{t-1}, would show evidence of a structural break at date T. Since the parameter a_{12} is a function of μ, he would also discover a structural break in a regression of p_t on m_{t-1}. The fact that this second break is missing in the UK money demand data caused *Favero* and *Hendry* to claim that the assumption of rational expectations is demonstrably false.

2. The claim is not that *all* examples of economic models lead one to expect to find structural breaks in money demand functions following a change in the money supply process. Rather, the money demand function that *Favero* and *Hendry* fit to the UK has this property. Their money demand function includes the expected inflation rate as an argument and it is the presence of this variable that allows them to use a super-exogeneity test for the *Lucas Critique*.

2.3 Learning as an Explanation of the Empirical Failure of RE Models

One possible explanation for the *Favero-Hendry* results is that rational expectations is a good characterization of 'normal periods' but the rational expectations assumption breaks down following a regime change. According to this argument, agents forecast the future using simple rules of thumb. They might, for example, guess that the future price is a linear function of the lagged money stock.

$$p_{t+1} = a_t + b_t m_{t-1}. \tag{7}$$

Initially, private agents might substitute arbitrary values of a_t and b_t into equation (1) and use these values to form a subjective expectation of p_{t+1}. Substituting this arbitrary expectation rule into equation (1) would lead to an actual law of motion for p_t that differs from the perceived law, equation (7). Repeated observations of the price would enable these agents to revise their forecast rules and eventually, one might hope that the parameters of the forecast rule would converge to the true values, described by equation (6). This, I believe, is a fair characterization of the work on learning by Albert MARCET and Thomas SARGENT [1989], and George EVANS and Seppo HONKAPOJHA [1990]. Although this agenda seems promising, I want to draw attention to an alternative explanation of the apparent failure of rational expectations models.

2.4 Indeterminacy as an Alternative Explanation

In a large class of RE models the assumptions of rational expectations and market clearing are insufficient to uniquely determine an equilibrium. Examples include the overlapping generations model, representative agent models, and models with money in the utility function or the production function.[3] There are many examples of models in this class in which the reduced form macro model leads to a set of equations of exactly the form of (1) and (2) with the one difference that the parameter restriction, $\alpha\mu < 1$ breaks down. These models have solutions that can be represented as VARs of the form:

$$p_{t+1} = \frac{\beta}{\alpha} m_t - \frac{1}{\alpha} p_t - \frac{1}{\alpha} u_t + w_{t+1}, \tag{8}$$

$$m_{t+1} = \mu m_t + v_{t+1}, \tag{9}$$

where w_{t+1} represents an arbitrary random variable with zero conditional mean. An economist observing data generated by this model would not be surprised by the *Favero-Hendry* results since in indeterminate rational expectations models the cross equation restrictions do not hold.

3. For a more extensive discussion see FARMER [1992].

3 How Do Agents Form Beliefs?

The indeterminacy approach has been widely applied to a range of phenomena including business cycles driven by animal spirits, FARMER and CUO [1994], underdevelopment in growth models, BENHABIB and PERLI [1994], and nominal rigidity in monetary models, BENHABIB and FARMER [1994]. But there is an important issue in these applications that has not been satisfactorily resolved. Advocates for models with indeterminate equilibria have not given a convincing explanation of the mechanism that enforces one equilibrium rather than another.

Indeterminacy does not occur in simple dynamic general equilibrium models with finite numbers of goods and agents. In infinite horizon examples one requires an externality of some kind. In overlapping generations models it is simpler to generate non-trivial examples although the demand and supply equations that arise in these examples typically violate standard assumptions on the signs of the slopes of demand and supply. These comments motivate the following example.

Consider the following model of a single market. In this model demand and supply depend on the current price and demand also contains a term in expected inflation that reflects the possibility of inventory speculation. The example is not derived from utility maximization and I present it simply to fix ideas. The example is described by the equations:

$$x_t^S = cp_t + b\left(p_t - p_{t+1}^E\right), \tag{10}$$

$$x_t^D = -dp_t, \tag{11}$$

in which x^S and x^D are supply and demand, p is the price and b,c and d are parameters. All variables are measured as deviations from the non-stochastic steady state. The rational expectation of this model satisfies the equation:

$$E_{t-1}\left[p_t\right] = \left(\frac{b}{b+c+d}\right) E_t\left[p_{t+1}\right],$$

and there exist multiple rational expectations equilibria whenever:

$$\left|\frac{b+c+d}{b}\right| < 1. \tag{12}$$

Condition (12) implies that either the supply curve must slope down ($c < 0$) or the demand curve must slope up ($d < 0$). Examples of both kinds can be found in the literature although for the purposes of this section I will assume that this condition holds, and I will not go further into a model that might deliver this result.

In much of the literature on indeterminate equilibria one proceeds by arguing that, if condition (12) is satisfied, there exist sunspot equilibria in the class:

$$p_{t+1} = \left(\frac{b+c+d}{b}\right) p_t + w_{t+1}, \tag{13}$$

where w_{t+1} is a 'sunspot' process with conditional mean zero. But if we adopt this argument then what is the mechanism in period t that causes the price to take one value rather than another? Agents cannot use the rational expectations price function as a forecast mechanism since if agents forecast with Equation (13) itself, the market will clear for *any* value of the current price. To see this, equate demand and supply from Equations (10) and (11) to give:

$$cp_t + b\left(p_t - p_{t+1}^E\right) = -dp_t. \tag{14}$$

The left side of this expression determines how suppliers in the market react to the current price and to their expectation of the future price. The right side determines market demand as a function of price. Suppose that suppliers form expectations using the expectation of p_{t+1} formed from Equation (13). Substituting:

$$E_t\left[p_{t+1}\right] = \left(\frac{b+c+d}{b}\right) p_t,$$

into Equation (14) gives the identity,

$$cp_t + b\left[p_t - \left(\frac{b+c+d}{b}\right) p_t\right] = -dp_t, \tag{15}$$

or,

$$0 = 0,$$

which is satisfied for any price. In other words, if agents use the rational expectations forecast mechanism to predict the future then the market clearing mechanism does not determine the price. The problem arises because there is no mechanism to anchor expectations. In the following section, I propose a resolution of the issue that I refer to as generalized adaptive expectations.

4 Generalized Adaptive Expectations

In this section, I propose a positive theory of expectations formation that can be used in indeterminate models to anchor the equilibrium. In section 5, I apply this criterion to a number of examples to show how it can be implemented in practice.

4.1 Adaptive and Rational Expectations

Consider the following class of linear rational expectations models:

(16) $$Y_t = A_1 Y_{t+1} + A_2 Y_{t+1}^E + e_{t+1}.$$

The term Y_t is an $n \times 1$ vector of variables, A_1 and A_2 are $n \times n$ non-singular matrices of parameters and e_{t+1} is an $n \times 1$ vector of fundamental errors that have expected values of zero conditional on information dated one period earlier. Y_{t+1}^E is the subjective expectation by agents of Y_{t+1}. The rational expectations assumption is that the subjective expectation of Y_{t+1} is given by:

(17) $$Y_{t+1}^E = E_t \left[Y_{t+1} | \Omega_t \right],$$

where Ω_t is the information set at date t. If one imposes this assumption, the model can be written as,

(18) $$Y_t = (A_1 + A_2) Y_{t+1} + A_3 V_{t+1},$$

where:

$$V_{t+1} = \begin{bmatrix} e_{t+1} \\ w_{t+1} \end{bmatrix},$$

$$w_{t+1} = E_t \left[Y_{t+1} | \Omega_t \right] - Y_{t+1},$$

and the elements of A_3 are given by:

$$A_3 = [I \quad A_2].$$

4.2 Solving the Model

The rational expectations solution to this class of models is found by solving the equation,

$$Y_t = (A_1 + A_2) Y_{t+1} + A_3 V_{t+1}.$$

4.2.1 The Determinate Case

It is well known that the solution is locally unique whenever the number of roots of $(A_1 + A_2)$ inside the unit circle equals the number of non predetermined variables. In this case, the solution is found by first computing:

$$Z_t = Q^{-1} Y_t,$$

where Q^{-1} is the inverse of the matrix of left eigenvectors of $(A_1 + A_2)$. Those elements of Z_t that correspond to roots of $(A_1 + A_2)$ inside the unit circle are set equal to zero, thus delivering the required number of additional boundary conditions to uniquely determine the solution to the *Markov* process,

$$Y_{t+1} = (A_1 + A_2)^{-1} \left(Y_t - A_3 V_{t+1} \right).$$

4.2.2 The Indeterminate Case and Generalized Adaptive Expectations

I will be concerned instead with the case in which all roots of $(A_1 + A_2)$ are outside the unit circle and where all of the elements of Y_t are non predetermined. In this case, one is free to choose values of w_{t+1} arbitrarily to generate sunspot solutions of the form:

$$Y_{t+1} = (A_1 + A_2)^{-1} \left(Y_t - A_3 V_{t+1} \right). \tag{19}$$

How is a solution of this kind implemented? Before the advent of rational expectations modeling it was typical to close a model with an assumption that determined how expectations were computed. In the case of rational expectations with multiple solutions, a similar assumption is required. Specifically let the model represented by the equation,

$$Y_t = A_1 Y_{t+1} + A_2 Y_{t+1}^E + e_{t+1}, \tag{20}$$

be supplemented by the expectations rule;

$$Y_{t+1}^E = B_1 Y_t^E + B_2 Y_t + B_3 V_t. \tag{21}$$

I will refer to a model in which expectations are generated by Equation (21) as a model that is closed by the assumption of *generalized adaptive expectations*.

In the case when Y_t is a scalar, $B_1 + B_2 = 1$ and $B_3 = 0$, these equations correspond to the standard form of adaptive expectations (introduced by Mark Nerlove [1958]; This assumption was frequently used in the 1960's to model expectations. According to the adaptive expectations hypothesis, agents enter period t with a given subjective expectation Y_t^E, formed at date $t - 1$. Adaptive expectations is equivalent to Equation (21) with the additional restriction that $B_1 + B_2 = 1$. With this additional restriction the equation implies that the subsequent period's expectation Y_{t+1}^E is formed by revising today's forecast by a fraction B_2 of today's forecast error. To see this note that when $B_1 + B_2 = 1$ then:

$$Y_{t+1}^E = Y_t^E + B_2 \left(Y_t - Y_t^E \right). \tag{22}$$

More generally, the adaptive expectations specification given in Equation (21) allows the subjective expectation of Y_{t+1}^E to depend on information contained in Y_t and information from previous periods as summarized by the state of subjective expectations, Y_t^E.

4.3. When Are Generalized Adaptive Expectations Rational?

Macroeconomists gave up on adaptive expectations because the parameters of the expectations rule were shown to depend on policy parameters. In the adaptive expectations tradition macroeconomic models were often built by appending 'error terms' to non-stochastic behavioral models. Rational expectations models, in contrast, are explicitly stochastic from the outset. The introduction of explicit theorizing about the stochastic structure of the economy was a big advance and an important contribution of rational expectations modeling.

But not all of the advances that came with rational expectations were progressive. For example, adaptive expectations models contained an explicit theory of how expectations are formed. Because the initial emphasis of rational expectations theorists was on models with unique equilibria, these models did not need a theory of expectations formation. In a rational expectations model with a unique equilibrium, the probability measure that describes subjective expectations of Y_{t+1} must coincide with the measure that describes realized values of Y_{t+1}. But in models with multiple rational expectations equilibria, the way that agents form beliefs is a critical part of the description of the economy since it is exactly this belief mechanism that decides which of many possible multiple equilibria will be implemented. The following proposition provides a practical way of pinning down an equilibrium in models with multiple rational expectations equilibria.

Proposition 1 *Consider the indeterminate rational expectations model supplemented with the generalized adaptive expectations rule, Equation (21). If the matrix B_1 is of full rank and if the matrices B_1, B_2 and B_3 satisfy the restrictions*:

(23) $$(B_1 + B_2) = (A_1 + A_2)^{-1},$$

(24) $$B_1^{-1} B_3 = -(A_1 + A_2)^{-1} A_3,$$

then the generalized adaptive expectations solution implements a rational expectations equilibrium.

Proof We first find an equation that determines the evolution of the rational expectation $E_t\left[Y_{t+1}\right]$ in a rational expectations equilibrium. If expectations are rational then, from (20), Y_t and $E_t\left[Y_{t+1}\right]$ are related by the equation:

(25) $$Y_t = (A_1 + A_2) E_t\left[Y_{t+1}\right].$$

To find the relationship between $E_{t-1}[Y_t]$ and $E_t\left[Y_{t+1}\right]$ subtract the expectation of Equation (19) from its realized value to give:

(26) $$Y_{t+1} = E_t\left[Y_{t+1}\right] - (A_1 + A_2)^{-1} A_3 V_{t+1}.$$

Since this equation holds at all dates,

$$Y_t = E_{t-1}[Y_t] - (A_1 + A_2)^{-1} A_3 V_t. \tag{27}$$

We seek a solution in which the rational expectation $E_{t-1}[Y_t]$ equals the subjective expectation Y_t^E. Imposing this assumption, substitute Equation (27) into Equation (25) to give:

$$E_{t-1}[Y_t] = (A_1 + A_2) E_t \left[Y_{t+1}\right] + (A_1 + A_2)^{-1} A_3 V_t.$$

Now suppose that Y_t^E is determined from the adaptive expectations formula, Equation (21). Substituting for Y_t in (21) from (25) gives:

$$Y_{t+1}^E = B_1 Y_t^E + B_2 (A_1 + A_2) Y_{t+1}^E + B_3 V_t, \tag{28}$$

or since, by assumption, B_1 is of full rank,

$$Y_t^E = B_1^{-1} (I - B_2 (A_1 + A_2)) Y_{t+1}^E - B_1^{-1} B_3 V_t. \tag{29}$$

Comparing Equations (27) and (29), it follows that the rational expectations solution is identical to the adaptive expectations solution if:

$$B_1^{-1} B_3 = -(A_1 + A_2)^{-1} A_3, \tag{30}$$

and,

$$B_1^{-1} (I - B_2 (A_1 + A_2)) = (A_1 + A_2). \tag{31}$$

Equation (30) is condition (24) of the proposition and Equation (31) is equivalent to condition (23). ■

5 A Series of Examples

This section illustrates how one would use generalized adaptive expectations to select an equilibrium in a series of examples. We begin with the micro market model from Section 3. This example has one dimensional dynamics and it provides an introduction to the method that can be easily compared with the use of the standard adaptive expectations approach.

5.1 The Micro Market Example

Consider the example of a micro market with indeterminate equilibria. Recall that this model is represented by the equations,

$$x_t^S = c p_t + b \left(p_t - p_{t+1}^E\right), \tag{32}$$

$$x_t^D = -d p_t. \tag{33}$$

The model can be written in the form of Equation (16) as follows:

$$p_t = (A_1 + A_2)\, p_{t+1} + A_3 V_{t+1},$$

where,

$$A_1 = 0, \quad A_2 = \frac{b}{b+c+d}, \quad A_3 = \frac{b}{b+c+d}, \quad V_{t+1} = w_{t+1}.$$

Using Proposition 1 we can find a generalized adaptive expectations rule of the form:

$$p^E_{t+1} = \lambda_1 p^E_t + \lambda_2 p_t + \lambda_3 V_t \tag{34}$$

and as long as the parameters of the rule satisfy the conditions,

$$\lambda_1 + \lambda_2 = \frac{b+c+d}{b}, \quad \frac{\lambda_3}{\lambda_1} = -1, \tag{35}$$

this adaptive expectations rule will implement a rational expectations equilibrium.

How does this rule provide an anchor to the current price? Consider the market clearing conditions:

$$bp^E_{t+1} = (b+c+d)\, p_t,$$

and substitute the generalized adaptive expectations rule (34) into this equation, to give:

$$b\left[\lambda_1 p^E_t + \left(\frac{b+c+d}{b} - \lambda_1\right) p_t - \lambda_1 w_t\right] = (b+c+d)\, p_t,$$

or,

$$p_t = p^E_t - w_t. \tag{36}$$

Equation (36) determines the current price in period t as a function of the expected price and the random shock. Contrast it with Equation (15), the identity that holds if agents use the rational expectations price function to forecast future prices. If agents forecast with the rational expectations price function, there is no anchor to pin down the price. In contrast, in the model with generalized adaptive expectations, the agents' subjective belief p^E_t provides such an anchor.

In the micromarket example, the expectation of next period's price evolves according to the equation:

$$p^E_{t+1} = \lambda_1 p^E_t + \left(\frac{b+c+d}{b} - \lambda_1\right) p_t - \lambda_1 w_t, \tag{37}$$

where w_t is an arbitrary sunspot variable. This example is special since the realization of p_{t+1} does not enter the market clearing equations (the matrix A_1 is equal to zero). For this example, we can use Equation (36) to show that the evolution of expectations is governed by the equation:

$$p^E_{t+1} = \left(\frac{b+c+d}{b}\right)\left(p^E_t - w_t\right), \tag{38}$$

and that this equation is independent of the value of λ_1.

5.2 The *Cagan* Model Revisited

Now consider the *Cagan* model with rational expectations. This example differs from the micro market model studied above by allowing for contemporaneous fundamental shocks to the demand and supply equations. The model has the following structure,

$$\begin{aligned} p_t &= \alpha E\left[p_{t+1}\right] + \beta m_t + u_t \\ m_t &= \mu m_{t-1} + v_t, \end{aligned}$$

where p_t is the price level, m_t is the money supply, α,β and μ are parameters and u_t and v_t are *i.i.d.* fundamental errors.

Since the *Cagan* model has a contemporaneous error term, u_t on the right hand side, we must define a new variable, z_t to write the model in the same form as Equation (18). Using the definition $z_t = p_t - u_t$ we can write the model in matrix form as follows:

$$\begin{bmatrix} 1 & -\beta \\ 0 & \mu \end{bmatrix}\begin{bmatrix} z_t \\ m_t \end{bmatrix} = \begin{bmatrix} \alpha & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} z_{t+1} \\ m_{t+1} \end{bmatrix} + \begin{bmatrix} 0 & \alpha \\ 1 & 0 \end{bmatrix}\begin{bmatrix} v_{t+1} \\ w_{t+1} \end{bmatrix},$$

where,

$$z_{t+1} = p_{t+1} - u_{t+1}, \qquad w_{t+1} = E_t\left[z_{t+1}\right] - z_{t+1}: \tag{39}$$

Or more compactly,

$$\begin{bmatrix} z_t \\ m_t \end{bmatrix} = (A_1 + A_2)\begin{bmatrix} z_{t+1} \\ m_{t+1} \end{bmatrix} + A_3\begin{bmatrix} v_{t+1} \\ w_{t+1} \end{bmatrix},$$

$$A_1 = \begin{bmatrix} 0 & \frac{\beta}{\mu} \\ 0 & \frac{1}{\mu} \end{bmatrix}, \qquad A_2 = \begin{bmatrix} \alpha & 0 \\ 0 & 0 \end{bmatrix}, \qquad A_3 = \begin{bmatrix} \frac{\beta}{\mu} & \alpha \\ \frac{1}{\mu} & 0 \end{bmatrix}.$$

The *Cagan* model has one predetermined variable, m_t and one non-predetermined variable, p_t. The roots of $(A_1 + A_2)$ are equal to α and $\frac{1}{\mu}$ and for a unique rational expectations equilibrium one requires that one of these roots is inside and one root outside the unit circle. Since $|\mu| < 1$ is a necessary condition for a stationary policy, uniqueness requires that $|\alpha| < 1$. This case is familiar and leads to the solution discussed in Section 4.2.1.

The interesting case for our purpose is when $|\mu| < 1$ and $|\alpha| > 1$, since in this case the model has multiple rational expectations equilibria corresponding to the solutions to the equation,

$$(40) \quad \begin{bmatrix} z_{t+1} \\ m_{t+1} \end{bmatrix} = (A_1 + A_2)^{-1} \begin{bmatrix} z_t \\ m_t \end{bmatrix} - (A_1 + A_2)^{-1} A_3 \begin{bmatrix} v_{t+1} \\ w_{t+1} \end{bmatrix}.$$

where w_{t+1} is generated by an arbitrary stochastic process with zero conditional mean.

Solutions of the form of Equation (40) have been known for some time. But how are they implemented? What is the economic process that causes agents to behave in one way rather than another ? To answer this question we will study how the model behaves if expectations are formed using generalized adaptive expectations. First, we write the model in the following way,

$$\begin{bmatrix} z_t \\ m_t \end{bmatrix} = \begin{bmatrix} 0 & \frac{\beta}{\mu} \\ 0 & \frac{1}{\mu} \end{bmatrix} \begin{bmatrix} z_{t+1} \\ m_{t+1} \end{bmatrix} + \begin{bmatrix} \alpha & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} z_{t+1}^E \\ m_{t+1}^E \end{bmatrix} + \begin{bmatrix} \frac{\beta}{\mu} & \alpha \\ \frac{1}{\mu} & 0 \end{bmatrix} \begin{bmatrix} v_{t+1} \\ w_{t+1} \end{bmatrix},$$

and let expectations be generated by the equation:

$$(41) \quad \begin{bmatrix} z_{t+1}^E \\ m_{t+1}^E \end{bmatrix} = B_1 \begin{bmatrix} z_t^E \\ m_t^E \end{bmatrix} + B_2 \begin{bmatrix} z_t \\ m_t \end{bmatrix} + B_3 \begin{bmatrix} v_t \\ w_t \end{bmatrix}.$$

Since Equation (41) is a direct application of generalized adaptive expectations we can apply Proposition 4.3 to establish that any adaptive process for which B_1, B_2, and B_3 satisfy the restrictions:

$$(42) \quad (B_1 + B_2) = \begin{pmatrix} \frac{1}{\alpha} & -\frac{\beta}{\alpha} \\ 0 & \mu \end{pmatrix},$$

and,

$$(43) \quad B_3 = -B_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

will implement a rational expectations equilibrium.

Suppose that agents use a generalized adaptive expectations rule of this kind. What does this say about their forecasts of prices? Recall that the price is equal to z_t plus a shock u_t. Price forecasts are found from the first row of Equation (41) and there are many rational forcast rules. Suppose we restrict ourselves to examples in which agents use only the realization of m_t to help forecast z_t (and not the $t-1$ expectation of m_t^E). This restriction implies that the B_1 matrix has the structure,

$$(44) \quad B_1 = \begin{pmatrix} \lambda_1 & 0 \\ b_{21} & b_{22} \end{pmatrix}$$

and hence,

$$B_3 = \begin{pmatrix} 0 & -\lambda_1 \\ -b_{22} & -b_{21} \end{pmatrix}. \tag{45}$$

In this case, we can read from the first row of Equation (41) that any adaptive process where:

$$z_{t+1}^E = \lambda_1 z_t^E + \lambda_2 z_t - \frac{\beta}{\alpha} m_t - \lambda_1 w_t, \tag{46}$$

and,

$$\lambda_1 + \lambda_2 = \frac{1}{\alpha},$$

will sustain a rational expectations equilibrium.[4] That is, the expectation of next period's price is formed as a weighted sum of last period's expectation, the current price, the value of the money supply and a sunspot shock w_t.

Generalized adaptive expectations differs from standard adaptive expectations in three ways. Firstly, agents directly update their expectation of the future price in response to shocks to the current money supply. Secondly, the rule allows for agents to factor a non-fundamental shock w_t into their forecast. Thirdly, the weights on expected and realized inflation sum to α^{-1} rather than to unity. Since, in practice, α is close to unity for the *Cagan* model this latter difference may be quantitatively unimportant. In other words, the adaptive expectations formula studied by *Cagan* is likely to be close to a rational expectations equilibrium during periods when the money supply process is stationary.

5.3 The Growth Model With Increasing Returns

In this section, I show how to apply generalized adaptive expectations to a version of the Real Business Cycle model developed by BENHABIB and FARMER [2000]. In their version of the real business cycle model, increasing returns to scale in technology cause the model to have multiple indeterminate rational expectations equilibria. The *Benhabib Farmer* model has the following structure,

$$\frac{1}{C_t} = E_t\left[\frac{1}{1+\rho}\frac{1}{C_{t+1}}\left(1 - \delta + \frac{aY_{t+1}}{K_{t+1}}\right)\right], \tag{47}$$

$$K_{t+1} = (1-\delta)\,K_t + Y_t - C_t, \tag{48}$$

$$Y_t = K_t^{\alpha} L_t^{\beta}, \tag{49}$$

4. To see this one only need substute the definition of z_t from Equation (39) into Equation (46) and verify that it satisfies Equation (41).

$$\frac{1}{C_t} = b\frac{Y_t}{L_t}. \tag{50}$$

These equations arise from a standard real business model in which a representative agent has logarithmic preferences over consumption and linear preferences over leisure. C_t is consumption, K_t is capital, Y_t is output and L_t is labor supply. ρ is the rate of time preference, δ the rate of depreciation, b is labor's share of income, a is capital's share and α and β are the elasticities of capital and labor in production.

When $b = \beta$ and $a = \alpha$ this model reduces to a standard real business cycle model. FARMER and GUO [1994] showed that if $\beta > 1$ and $\alpha > a$ the model has indeterminate equilibria driven by sunspots. Once again, in the sunspot economy, the question arises as to what implements a particular sunspot equilibrium. The following analysis uses proposition 1 to implement an equilibrium using adaptive expectations.

For the real business cycle model one first log linearizes each equation to give:

$$-c_t = -c^E_{t+1} + a_1 y^E_{t+1} - a_1 k^E_{t+1}, \tag{51}$$

$$k_{t+1} = a_2 k_t + a_3 y_t + a_4 c_t, \tag{52}$$

$$y_t = \alpha k_t + \beta l_t, \tag{53}$$

$$-c_t = y_t - l_t, \tag{54}$$

where a_1, a_2, a_3, are a_4 compound parameters that come from the *Taylor Series* approximation of these equations and the lower case letters refer to logarithmic deviations from the non-stochastic steady state. By substituting the static Equations, (53) and (54), into the dynamic Equations, (51) and (52), one arrives at a model in the form of Equation (18);

$$\begin{bmatrix} c_t \\ k_t \end{bmatrix} = A_1 \begin{bmatrix} c_{t+1} \\ k_{t+1} \end{bmatrix} + A_2 \begin{bmatrix} c^E_{t+1} \\ k^E_{t+1} \end{bmatrix} + A_3 V_{t+1},$$

$$V_{t+1} = w_{t+1} = E_t\left[c_{t+1}\right] - c_{t+1}.$$

This example is more complicated than the previous two cases since in general the matrix:

$$(A_1 + A_2)^{-1}$$

has non zero elements in every position. This implies that if agents use the generalized adaptive expectations mechanism,

$$\begin{bmatrix} c^E_{t+1} \\ k^E_{t+1} \end{bmatrix} = B_1 \begin{bmatrix} c_{t+1} \\ k_{t+1} \end{bmatrix} + B_2 \begin{bmatrix} c^E_{t+1} \\ k^E_{t+1} \end{bmatrix} + B_3 V_t,$$

then the expectations rule adds four new parameters; the elements of B_1. These four parameters determine the weights with which fundamental and non-fundamental shocks enter the forecasts of k_{t+1} and c_{t+1} conditional on observing c_t and k_t.

6 Evaluating the Alternatives

This paper began with the failure of standard rational expectations models identified by *Favero* and *Hendry*. The two explanations that I have given for the *Favero* and *Hendry* results are spawned by alternative research strategies. One strategy is to insist that sensible models have unique equilibria. It is this route that leads to learning. The second is to relax the insistence on uniqueness and to add a theory of beliefs as a selection device. The latter route may have room for a theory of learning to select an equilibrium, but it does not have room for learning as part of the explanation of the dynamic behavior of the data. In models with indeterminacy, all learning has already taken place. How do these alternative theories explain the *Favero* and *Hendry* results?

According to the learning hypothesis, agents are continually updating the weight that they give to variables that might be relevant in forecasting future prices. Expectations are not rational, but the mechanism that generates expectations is sufficiently sophisticated that, in response to changes of policy from one simple rule to another, the economy quickly settles down to a new rational expectations equilibrium. In contrast, according to the indeterminacy hypothesis, agents have already learned the parameters of the correct forecasting rule and these parameters are invariant to changes in the policy regime.

Although both explanations can potentially account for the evidence, my own research has focused on the latter. There are two reasons. First, it is a formidable challenge to model learning rules that are sophisticated enough to capture human behavior. It seems likely that any plausible rule might fit data well within sample, but would not forecast well out of sample. If one estimates a learning rule that works well for regimes of constant money growth, the same rule is unlikely to work well when the Fed switches to an inflationary policy. Although this is not a reason to give up on learning; it suggests that an explanation of any particular episode of a regime change can be analyzed only after it has occurred.

My second reason for favoring the indeterminacy hypothesis is that it explains episodes of regime changes and normal functioning of the economy with a single theory. It offers the explanation that the real effects of monetary shocks are caused by equilibrium adjustments to a correctly perceived change in the future path of the real interest rate. Models that invoke the learning hypothesis, in contrast, must add a theory of nominal rigidity. One is left with the problem of identifying which episodes are normal and which are episodes of regime shifts.

7 Anything Goes?

Although I have argued that learning may not be the correct explanation of the *Favero-Hendry* results, this does not mean that theories of learning should be abandoned. There is a second avenue that the learning literature has followed. As a selection device in models with multiple rational expectations equilibria. Initially, some authors had hoped that one might argue that a plausible learning mechanism could isolate a unique equilibrium. But work by Michael WOODFORD [1990] has shown that plausible learning mechanisms can converge to sunspot equilibria and John DUFFY [1994] has shown that they may converge to one of a set of indeterminate equilibria. Although this work is in its early stages, it is possible that a theory of learning may help us to understand which of many possible equilibria one observes in data. If alternative theories of learning select different equilibria, one might test one theory against another by checking to see which equilibrium best characterizes the data. In models with indeterminate equilibria, different rational expectations equilibria impose different restrictions on the covariance properties of the data. It is true that they place one less restriction than regular rational expectations models, but freeing this restriction by adding an additional parameter is exactly what the failure of the *Favero-Hendry* test requires.

If one accepts the idea that indeterminacy may help us to understand macroeconomic data, how should our research progress? First, one needs to develop small dynamic general equilibrium models that are consistent with the evidence. If the data rejects the cross equation restrictions associated with a regular model then the first amendment should be to choose parameters that are consistent with the absence of these restrictions in the theoretical model. In other words, a model with an indeterminate equilibrium. Second, one needs to ask, which of the many possible equilibria best fits the data. Jang Ting GUO and I have shown in other work, FARMER and GUO [1995] that this question comes down to estimating the covariance matrix of a set of shocks that drives the reduced form of a dynamic G.E. model. Given that one has isolated a particular equilibrium, one is led to ask why this is the equilibrium that we observe. Here, there is a role for theories of learning and one might hope that there will be a coincidence of theory and fact. One would like to be able to explain the particular equilibrium as the fixed point of a learning mechanism of the kind studied in the literature on disequilibrium learning rules. ▼

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