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WHAT WE DON'T KNOW ABOUT THE MONETARY TRANSMISSION MECHANISM AND WHY WE DON'T KNOW IT

Andreas Beyer
European Central Bank

Roger E.A. Farmer
University of California, Los Angeles

We study identification in a class of linear rational expectations models. For any given exactly identified model, we provide an algorithm that generates a class of equivalent models that have the same reduced form. We use our algorithm to show that a model proposed by Jess Benhabib and Roger Farmer is observationally equivalent to the standard new-Keynesian model when observed over a single policy regime. However, the two models have *different* implications for the design of an optimal policy rule.

Keywords: Rational Expectations, Identification, New-Keynesian Model, Benhabib–Farmer Model

1. INTRODUCTION

> It is my view, however, that rational expectations is more deeply subversive of identification than has yet been recognized.
>
> Christopher A. Sims, *Macroeconomics and Reality*, 1980, p. 7.

This quotation is now 25 years old, but it has weathered well. It appeared in a paper that introduced vector autoregressions as an alternative to structural models at a time when the rational expectations agenda was in its infancy. A quarter of a century later, applied macroeconomists continue to estimate structural equations

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without paying careful attention to the identifying assumptions that one requires for a particular equation to make sense.

One popular approach to estimation of an equation that includes expectations of future variables is to replace the expectations by their realized values and to estimate the model using instrumental variables. This method, first discussed by McCallum (1976), has been widely used in recent work on applied monetary economics to estimate the parameters of one or more equations in a new-Keynesian model of the monetary transmission mechanism.[1] Although it is possible to estimate a single equation using instruments, the assumptions that are necessary to make any particular identification valid in the context of a complete structural model are rarely spelled out.[2] In this paper we show that the new-Keynesian identifying assumptions are at best untestable, and we provide a credible alternative identification scheme that provides a different answer to an important policy question: Should monetary policy be active or passive?

Our paper is organized as follows. Section 2 introduces a class of linear rational expectations models and defines the concepts of observational equivalence and identification. Section 3 contains our main example. We present a new-Keynesian model and show that an alternative explicit microeconomic theory of the monetary transmission mechanism due to Benhabib and Farmer (2000) has the same reduced form. This is a problem for the policy maker because the two observationally equivalent models have different determinacy properties and, therefore, different policy implications. In Section 4 we present the algorithm that we used to construct this example. Section 5 wraps up with a short conclusion.

2. IDENTIFICATION AND OBSERVATIONAL EQUIVALENCE IN RATIONAL EXPECTATIONS MODELS

We begin with a brief review of some definitions and basic concepts. Our discussion of identification is based on Rothenberg (1971) and an excellent survey of this and related concepts can be found in Hsiao (1983). Skepticism of the ability of economic theory to deliver a credible set of identifying restrictions can be traced back to Liu (1960) and, in the context of rational expectations models, to Pesaran (1987) and Sims (1980).

2.1. Observational Equivalence

We take Y to be a vector-valued random variable that takes values in R^l. Y has a probability distribution function that belongs to a known family of distributions $\mathcal{F}$ on R^l. A structure S is a set of hypotheses that implies a unique distribution function $F(S) \in \mathcal{F}$. A set of structures S is called a model and by definition there is a unique distribution function associated with each S in S. The following definitions are due to Rothenberg (1971, p. 578).

DEFINITION 1 (Rothenberg). *Two structures in* S *are* observationally equivalent *if they imply the same probability distribution for the random variables* Y.

DEFINITION 2 (Rothenberg). *A structure S in* S *is said to be* identifiable *if there is no other structure in* S *that is observationally equivalent.*

Definitions (1) and (2) apply to very general classes of models. In the following section we apply them to a class of linear rational expectations models.

2.2. Rational Expectations

We will be concerned with models of the form

$$AY_t + FE_t[Y_{t+1}] = B_1 Y_{t-1} + B_2 E_{t-1}[Y_t] + C + \Psi_v V_t, \tag{1}$$

$$E_t[V_t V_s'] = \begin{cases} I_l, & t = s, \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

A, F, Ψ_v, B_1, and B_2 are $l \times l$ matrices of coefficients, C is an $l \times 1$ matrix of constants, E_t is a conditional expectations operator, and $\{V_t\}$ is a weakly stationary i.i.d. stochastic process with covariance matrix Ω_{vv} and mean zero. Lowercase letters are scalars, and uppercase letters represent vectors or matrices. We maintain the convention that endogenous variables appear on the left side of each equation and explanatory variables appear on the right. Our definition of a structure includes equations (1) and (2) together with the additional assumptions that the shocks V_t are i.i.d.

Equation (1) is a system of l equations in $2l$ endogenous variables $\{Y_t, E_t[Y_{t+1}]\}$. To close the model one requires additional equations. Under the rational expectations assumption these are provided by the following definition of the nonfundamental errors,

$$W_t = Y_t - E_{t-1}[Y_t], \tag{3}$$

plus the assumption that

$$\lim_{T\to\infty} E[Y_T] < \infty. \tag{4}$$

2.3. The Canonical Form

Combining equations (1) and (3), we arrive at the following representation of a structural linear rational expectations model, which Sims (2002) refers to as the *canonical form*:

$$\underbrace{\begin{bmatrix} A & F \\ I & 0 \end{bmatrix}}_{\tilde{A}_0}\underbrace{\begin{bmatrix} Y_t \\ E_t[Y_{t+1}] \end{bmatrix}}_{X_t} = \underbrace{\begin{bmatrix} B_1 & B_2 \\ 0 & I \end{bmatrix}}_{\tilde{A}_1}\underbrace{\begin{bmatrix} Y_{t-1} \\ E_{t-1}[Y_t] \end{bmatrix}}_{X_{t-1}} + \underbrace{\begin{bmatrix} C \\ 0 \end{bmatrix}}_{\tilde{C}} + \underbrace{\begin{bmatrix} \Psi_v \\ 0 \end{bmatrix}}_{\tilde{\Psi}_v} V_t + \underbrace{\begin{bmatrix} 0 \\ I \end{bmatrix}}_{\tilde{\Psi}_w} W_t. \tag{5}$$

We can write equation (5) more compactly as follows:

$$\tilde{A}_0 X_t = \tilde{A}_1 X_{t-1} + \tilde{C} + \tilde{\Psi}_v V_t + \tilde{\Psi}_w W_t. \quad \textbf{(6)}$$

Equation (6) is similar to the class of structural models considered by the Cowles Commission.[3] It differs by adding a set of nonfundamental error terms, W_t, and requiring that the expected value of Y_t remain bounded. The error terms, W_t, are different from the shocks that drive a Cowles Commission model, since some or all of them may be endogenously determined as part of the solution of the model.

2.4. The Reduced Form

The reduced form of an econometric model is a set of equations that explains each endogenous variable as a function of exogenous and predetermined variables. The reduced form of equation (1) is given by the equation

$$X_t = \Gamma^* X_{t-1} + C^* + e_t, \quad \textbf{(7)}$$

where the reduced form residuals e_t are functions of the fundamental and nonfundamental shocks,

$$e_t = \Psi_v^* V_t + \Psi_w^* W_t. \quad \textbf{(8)}$$

In the case of a unique equilibrium, Ψ_w^* is identically zero and, in this case, only the fundamental shocks influence the behavior of the system.

2.5. The Dynamics of the Reduced Form

The reduced form governs the behavior of the state variables Y_t and their expectations $E_t[Y_{t+1}]$. In computing the reduced form, there are three possible cases to consider: (1) there is a unique equilibrium, (2) there are multiple stationary indeterminate equilibria, or (3) no stationary equilibrium exists. In the following paragraphs we discuss cases (1) and (2).

In almost all cases, the reduced form parameter matrix Γ^* has reduced rank and it is possible to partition X_t into two disjoint subsets $X_t = (X_{1t}, X_{2t})$ such that X_{1t} is described by a VAR(1),

$$X_{1t} = \Gamma_{11}^* X_{1t-1} + C_1^* + e_{1t}, \quad \textbf{(9)}$$

$$e_{1t} = \Psi_{1v}^* V_t + \Psi_{1w}^* W_{1t}, \quad \textbf{(10)}$$

and X_{2t} is an affine function of X_{1t},

$$X_{2t} = C_2^* + M^* X_{1t}. \quad \textbf{(11)}$$

The one exception to this rule is when the equilibrium is indeterminate and the degree of indeterminacy is equal to l. In this case the matrix Γ^* has full rank and X_{2t} is empty.

In the familiar case of a unique equilibrium the number of unstable generalized eigenvalues of $\{\tilde{A}_0, \tilde{A}_1\}$ is equal to l.[4] In this case one can choose $X_{1t} = Y_t$ and equation (9) has the form

$$\begin{aligned} Y_t &= \Gamma_{11}^* Y_{t-1} + C_1^* + e_{1t}, \\ e_{1t} &= \Psi_{1v}^* V_t. \end{aligned} \tag{12}$$

When the equilibrium is unique, the shocks W_t do not enter the reduced form and in that case X_{2t} is equal to $E_t\,[Y_{t+1}]$, equation (**11**) takes the form

$$E_t\,[Y_{t+1}] = C_2^* + M^* Y_t, \tag{13}$$

and M^* and Γ_{11}^* are $l \times l$ matrices of full rank.

If the number of unstable generalized eigenvalues is less than l, the solution is said to be indeterminate. The degree of indeterminacy, r, is equal to $l - n$, where l is the dimension of Y_t and n is the number of unstable roots; r can vary between 1 and l. Although, in this case, it will still be possible to partition X_t and write the reduced form as a VAR(1), it may not be possible to choose this partition in a way that excludes $E_t[Y_{t+1}]$ from X_{1t}.

Our definition assumes that every structure is associated with a unique probability distribution for the observable variables. If the solution to a linear rational expectations model is nonunique, we take the view that the set of hypotheses that define the structure is incomplete and the economist must add a probability model for one or more of the nonfundamental shocks W_t. If there are r degrees of indeterminacy then one may proceed by partitioning W_t into two disjoint subsets, $W_{1t} \in R^r$, $W_{2t} \in R^{l-r}$, and making the assumption that

$$E_t[W_{1t} W_{1s}'] = \begin{cases} I_r, & t = s, \\ 0, & \text{otherwise.} \end{cases}$$

A complete model must then add restrictions to the elements of Ψ_v and Ψ_w that determine how the fundamental shocks and the r elements of W_{1t} interact with the structure. This approach amounts to reclassifying r of the nonfundamental shocks as new fundamentals.[5]

3. IDENTIFICATION IN THE NEW-KEYNESIAN MODEL

In this section we provide an example that illustrates our main result. We show that within the class of linear rational expectations models there exist examples of structures with different microfoundations that are observationally equivalent. One of these structures is driven by fundamentals alone; the other is driven in part by nonfundamental "sunspot" shocks. Unlike previous examples of observational equivalence of the kind discussed by Sargent (1976), the structures we present in this section have different determinacy properties.[6]

Recall that a structure is a set of hypotheses that implies a unique distribution function $F(S) \in \mathbb{F}$. A model is a set of structures. Our exercise is to refine the set of hypotheses that define the linear rational expectatations model in two different ways. The first exactly identifies the new-Keynesian model. The second exactly identifies a microfounded model due to Benhabib and Farmer 2000. Each model is exactly identified, but the models are nonnested and they each generate the same unique distribution function $F(S) \in \mathbb{F}$.

3.1. Two Alternative Models

Our first model is based on a new-Keynesian theory of aggregate supply. In this theory money has real effects because some agents are unable to adjust prices in every period. Our second model is based on the theory of aggregate supply outlined in Benhabib and Farmer (2000). In this theory money has real effects either because it is useful in production or because real balances influence labor supply.

The following equations represent a parameterized three-equation version of the new-Keynesian model:

$$y_t + a_{13}(i_t - E_t[\pi_{t+1}]) + f_{11}E_t[y_{t+1}] = b_{11}y_{t-1} + c_1 + v_{1t}, \tag{14}$$

$$a_{21}y_t + \pi_t + f_{22}E_t[\pi_{t+1}] = c_2 + b_{22}\pi_{t-1} + v_{2t}, \tag{15}$$

$$i_t + f_{32}E_t[\pi_{t+1}] = b_{33}i_{t-1} + c_3 + v_{3t}. \tag{16}$$

In our notation $[a_{ij}]$, $[f_{ij}]$, and $[b_{ij}]$ represent the coefficients of variable j in equation i on contemporaneous endogenous variables, expected future variables, and lagged endogenous variables. y_t is the output gap, i_t is the fed funds rate, π_t is inflation, and v_{1t}, v_{2t}, and v_{3t} are fundamental shocks to the equations of the model. c_i is the constant in equation i.

Equation (14) is an "optimizing IS curve," equation (15) is a new-Keynesian Phillips curve, and (16) is a central bank reaction function. A model of this kind has been widely used to model the inflation process in a closed economy [Clarida et al. (2000), Galí and Gertler (1999), Lindé (2001, 2005), and Rotemberg and Woodford (1998)] and a modified version of the model has been used to study inflation dynamics in open economies [Clarida et al. (2002)].

To parameterize the "true model" we chose parameters similar to those that have been estimated by Lubik and Schorfheide (2004), Ireland (2004), and Beyer et al. (2005). Beyer et al. provide a detailed discussion of the properties of this model under alternative estimation schemes and Beyer and Farmer (2004b) derive the implications of the restricted estimates for impulse responses to alternative shocks. Table 1 contains our specification of the new-Keynesian data-generating process (DGP).

Our alternative model, based on Benhabib and Farmer (2000), is represented in equations (17)–(19). Equation (17) is identical to the optimizing IS curve in

TABLE 1. Parameters of the NK data generation process

	Euler equation normalized for y_t			
Var.	$i_t - E_t\,[\pi_{t+1}]$	$E_t[y_{t+1}]$	y_{t-1}	Constant
Name	a_{13}	f_{11}	b_{11}	c_1
	0.05	−0.5	0.50	0.0015
	Phillips curve normalized for π_t			
Var.	y_t	$E_t[\pi_{t+1}]$	π_{t-1}	Constant
Name	a_{21}	f_{22}	b_{22}	c_2
	−0.5	−0.8	0.25	−0.0010
	Policy rule normalized for i_t			
Var.	y_t	$E_t[\pi_{t+1}]$	i_{t-1}	Constant
Name	a_{31}	f_{32}	b_{33}	c_3
	−0.5	−1.1	0.8	−0.012

the new-Keynesian model; equations (18) and (19) are different from their new-Keynesian counterparts

$$y_t + a_{13}\,(i_t - E_t\,[\pi_{t+1}]) + f_{11}E_t\,[y_{t+1}] = b_{11}y_{t-1} + c_1 + \bar{v}_{1t}, \qquad \mathbf{(17)}$$

$$y_t + \bar{a}_{23}i_t = \bar{b}_{21}y_{t-1} + \bar{b}_{23}i_{t-1} + \bar{c}_2 + \bar{v}_{2t}, \qquad \mathbf{(18)}$$

$$\bar{a}_{31}y_t + \bar{a}_{32}\pi_t + i_t = \bar{b}_{33}i_{t-1} + \bar{c}_3 + \bar{v}_{3t}. \qquad \mathbf{(19)}$$

Equation (18) is the Benhabib–Farmer theory of aggregate supply by which a higher value of the nominal interest rate causes firms and households to economize on real balances. Because real balances are productive inputs to the real economy, a reduction in real balances causes a loss of output. Benhabib and Farmer provide a theory that explains how this effect can be large even when the share of resources attributed to money as a productive asset is small. We have allowed for a propagation mechanism in this equation by including the lagged output gap and lagged nominal interest rate as additional variables.

Equation (19) is the policy rule. This differs from our new-Keynesian representation of policy in one respect; we have assumed that the Fed responds to current inflation instead of to expected future inflation. This variation is important because we are searching for a version of the Benhabib–Farmer model that is observationally equivalent to the new-Keynesian model. The Benhabib–Farmer aggregate supply curve does not depend on inflation and, because inflation appears contemporaneously in the new-Keynesian model, the Benhabib–Farmer model must introduce this variable elsewhere in the system if the two structural models are to have the same reduced form.

To find parameterizations of an alternative model that has the same reduced form, we used the algorithm described in Section 4. Table 2 reports the values of the structural parameters of the alternative model. The most important feature of the differences between these models is that the Benhabib–Farmer model is indeterminate and may be driven, in part, by sunspot shocks.

TABLE 2. Equivalent parameters of the Benhabib–Farmer model

	Euler equation, normalized for y_t			
Var.	$i_t - E_t[\pi_{t+1}]$	$E_t[y_{t+1}]$	y_{t-1}	Constant
Name	a_{13}	f_{11}	b_{11}	c_1
	0.05	−0.5	0.50	0.0015
	Supply curve normalized for y_t			
Var.	i_t	y_{t-1}	i_{t-1}	Constant
Name	$\bar{a}_{23}$	$\bar{b}_{21}$	$\bar{b}_{23}$	$\bar{c}_2$
	0.09	0.74	−0.04	0.0064
	Policy rule normalized for i_t			
Var.	y_t	π_t	i_{t-1}	Constant
Name	$\bar{a}_{31}$	$\bar{a}_{32}$	$\bar{b}_{33}$	$\bar{c}_3$
	−1.02	−0.26	0.63	0.0132

The true new-Keynesian model has the reduced form

$$X_t = \Gamma^* X_{t-1} + C^* + \Psi_v^* V_t,$$

where $X_t = (Y_t, E_{t-1}[Y_t])$, whereas the equivalent Benhabib–Farmer model has a reduced form,

$$X_t = \Gamma^* X_{t-1} + C^* + \bar{\Psi}_v^* \bar{V}_t + \bar{\Psi}_{w1}^* \bar{W}_{1t}.$$

We checked that the reduced form parameters $\{\Gamma^*(\theta), C^*(\theta)\}$ are indeed equal to those of the equivalent model, $\{\Gamma^*(\bar{\theta}), C^*(\bar{\theta})\}$, and, using the algorithm from Section 4, we computed a variance–covariance matrix $\bar{\Omega}_1$ such that

$$\Psi_v^* I_l \Psi_v^{*\prime} = [\bar{\Psi}_v^* \bar{\Psi}_{w1}^*] \bar{\Omega}_1 [\bar{\Psi}_v^* \bar{\Psi}_{w1}^*]'.$$

The shocks V_t and $(\bar{V}_t, \bar{W}_{1t})$ that drive the two models are observationally equivalent.

3.2. Comparative Dynamics of the Two Models

Table 3 presents a comparison of the generalized eigenvalues of the true model and the Benhabib–Farmer equivalent model arranged in descending order of absolute value. Stable roots are in boldface. The true model has three unstable roots,

TABLE 3. A comparison of the roots of the two models

	1st	2nd	3rd	4th	5th	6th
True model	∞	1.39	1.39	**0.33**	**0.62**	**0.62**
Equivalent model	∞	∞	**0**	**0.33**	**0.62**	**0.62**

leading to a unique determinate equilibrium. The equivalent model has the same three stable roots as the true model, but one of the unstable roots is replaced by a generalized eigenvalue of zero.

The occurrence of an extra zero eigenvalue in the equivalent model implies that there is one degree of indeterminacy in the way the system responds to fundamental shocks. In any given period, contemporaneous fluctuations in output, the interest rate, and inflation might in part be due to self-fulfilling beliefs.

3.3. Policy Implications of Observational Equivalence

A number of authors have taken up the issue of optimal policy in the new-Keynesian model. Woodford (2003) has argued that the central bank should strive to implement a policy that leads to a unique determinate rational expectations equilibrium because, if policy admits the possibility of indeterminacy, nonfundamental shocks may contribute to the variance of inflation and unemployment. This consideration suggests that a policy maker that dislikes variance should pick a policy rule that leads to a determinate equilibrium.

In a simple version of the new-Keynesian model equilibrium is determinate if the central bank responds to expected inflation by increasing the real interest rate and it is indeterminate if it responds by lowering it. In the former case, the central bank increases the nominal interest rate by more than one-for-one if it expects additional future inflation; a policy with this property is said to be *active*. In the latter case the central bank increases the interest rate by less than one-for-one if it expects additional inflation; in this case the policy is said to be *passive*.

In contrast, in a simple version of the Benhabib–Farmer model, equilibrium is determinate when the Fed follows a passive monetary policy. Our work suggests that an econometrician, by observing data from a period in which policy followed a stable rule, cannot tell whether the policy followed by the Fed led to a determinate or an indeterminate equilibrium.

4. AN ALGORITHM TO FIND CLASSES OF EQUIVALENT MODELS USING LINEAR RESTRICTIONS

In this section we provide an algorithm (implemented in Matlab as `FindEquiv`) to construct equivalence classes of structural models that have the same reduced form. For computational reasons we begin with a determinate model. This assumption is unrestrictive because our purpose is to establish, by means of an example, that there may exist determinate and indeterminate models that are observationally equivalent.

4.1. Structural and Reduced Form Parameters Defined

Consider a structural model given by equation (5) and define the vector of structural parameters

$$\theta = \text{vec}\,[(A, F, B_1, B_2, C, \Psi_v)'].$$

We refer to θ as the *true parameters* and to equation (5) as the *true* model. The assumption that the covariance matrix of V_t is the identity matrix is unrestrictive because we allow for correlated shocks to the structural equations through the impact matrix Ψ_v.

The reduced form of equation (5) is represented by equations (7) and (8) and is parameterized by the vector

$$\phi(\theta) = \operatorname{vec}[(\Gamma^*, C^*, \Psi_v^*)'].$$

By assumption, we begin with a determinate model and so the parameters Ψ_w^* that appear in (8) are identically zero. Our notation reflects the functional dependence of ϕ on θ. We refer to ϕ as the *reduced form* parameters.

Our next step is to forget that we know the true model and to trace the steps that would be followed by an econometrician who had access to an infinite sequence of data generated by the model and who used this data to recover the reduced form parameters ϕ. The econometrician combines his estimated reduced form with an economic theory and recovers some possibly different model that we call $\bar{\theta}$.

Following Fisher (1966), our econometrician establishes a set of linear equations linking ϕ to the structural parameters in his model, $\bar{\theta}$. He adds a set of linear restrictions of the form $R\bar{\theta} = r$ and solves the resulting linear equation system for $\bar{\theta}$ as a function of ϕ, r, and R.

Let the structural model of the econometrician be denoted as

$$\begin{bmatrix} \bar{A} & \bar{F} \\ I & 0 \end{bmatrix} \begin{bmatrix} Y_t \\ E_t[Y_{t+1}] \end{bmatrix} = \begin{bmatrix} \bar{B}_1 & \bar{B}_2 \\ 0 & I \end{bmatrix} \begin{bmatrix} Y_{t-1} \\ E_{t-1}[Y_t] \end{bmatrix} + \begin{bmatrix} \bar{C} \\ 0 \end{bmatrix} + \begin{bmatrix} \bar{\Psi}_v \\ 0 \end{bmatrix} \bar{V}_t + \begin{bmatrix} 0 \\ I \end{bmatrix} \bar{W}_t. \tag{20}$$

We refer to $\bar{\theta}$ as the *equivalent parameters* and to equation (20) as the *equivalent model*. Premultiplying (7) by $[\bar{A}\ \bar{F}]$ and equating coefficients leads to the matrix equation

$$\underset{l\times 2l}{\begin{bmatrix} \underset{l\times l}{\bar{A}} & \underset{l\times l}{\bar{F}} \end{bmatrix}} \underset{2l\times(3l+1)}{\begin{bmatrix} \underset{2l\times 2l}{\Gamma^*} & \underset{2l\times 1}{C^*} & \underset{2l\times l}{\Psi_v^*} \end{bmatrix}} = \underset{l\times(3l+1)}{\begin{bmatrix} \underset{l\times l}{\bar{B}_1} & \underset{l\times l}{\bar{B}_2} & \underset{l\times l}{\bar{C}} & \underset{l\times l}{\bar{\Psi}_v} \end{bmatrix}}. \tag{21}$$

After equation (21) is rearranged and the properties of the Kronecker product are exploited, this system can be written as the following set of $l(3l+1)$ equations in the $l(5l+1)$ parameter vector $\bar{\theta}$:

$$\underset{l(3l+1)\times l(5l+1)}{H(\phi)} \underset{l(5l+1)\times 1}{\bar{\theta}} = \underset{l(3l+1)\times 1}{h}. \tag{22}$$

The details of this construction are given in the Appendix.

To recover a unique vector $\bar{\theta}$ that satisfies these equations, we require an additional $2l^2$ independent linear restrictions, which we assume are given by economic theory in the form of exclusion restrictions or as linear constraints. We parameterize these restrictions with a matrix R and a vector r such that

$$\underset{l(2l)\times l(5l+1)}{R}\ \underset{l(5l+1)\times 1}{\bar{\theta}} = \underset{l(2l)\times 1}{r}. \qquad (23)$$

Stacking equations (22) and (23) leads to the system

$$\underset{l(5l+1)\times l(5l+1)}{J}\ \underset{l(5l+1)}{\bar{\theta}} = \underset{l(5l+1)\times 1}{r}, \qquad (24)$$

where

$$J = \begin{bmatrix} \underset{(3l+1)\times(5l+1)}{H} \\ \underset{l(2l)\times l(5l+1)}{R} \end{bmatrix} \quad \text{and} \quad j = \begin{bmatrix} \underset{l(3l+1)\times 1}{h} \\ \underset{l(2l)\times 1}{r} \end{bmatrix}.$$

For the structural model to be identified, the matrix J must have full rank and the rows of equation (23) must identify *different* structural equations. This requires that the rank and the order conditions [Fisher (1966)] must be checked for each equation of the system. When identification is satisfied, the econometrician can recover the equivalent model $\bar{\theta}$ from the estimates of the reduced form (contained in ϕ) and the restrictions contained in (23). By construction, $\bar{\theta}$ is observationally equivalent to the true model θ and both models lead to the same reduced form; that is,

$$\phi(\theta) = \phi(\bar{\theta}).$$

The restriction matrix R that was used to compute the example in first part of the paper is available in the Matlab file NKexample.m (see Note 6).

4.2. Equivalent Representations of the Solution

To generate an equivalent model, the user need only follow the steps contained in Section 4.1. However, when this procedure is followed and the reduced form is computed using `SysSolve` or an equivalent program such as Sim's algorithm `GENSYS`, the resulting reduced form will typically look very different from that of the original model. However, these reduced forms are in fact equivalent; they just use different sets of state variables that span the same state space. This section explains how the user can verify the equivalence of the two reduced forms.

Let equation (25) represent the reduced form of a model that has a unique equilibrium,

$$\begin{aligned} Y_t &= \Gamma_{11}^* X_{t-1} + C_1^* + \Psi_v^* V_t, \\ E_t[Y_{t+1}] &= C_2^* + M^* Y_t. \end{aligned} \qquad (25)$$

We assume that the econometrician identifies an equivalent model that has an indeterminate equilibrium and we write the reduced form of this model as follows:

$$\begin{aligned} X_{1t} &= \bar{\Gamma}^*_{11} X_{1t-1} + \bar{C}^*_1 + \bar{\Psi}^*_v \bar{V}_t + \bar{\Psi}^*_w \bar{W}_t, \\ X_{2t} &= \bar{C}^*_2 + \bar{M}^* X_{1t}. \end{aligned} \tag{26}$$

The algorithm we use to generate an equivalent model does not always choose a representation of the reduced form for which $X_{1t} = Y_t$. To establish observational equivalence we use a second algorithm, implemented in Matlab as `convert`, to rewrite the equivalent model using Y_t as the state variables. This leads to the representation

$$\begin{aligned} Y_t &= \bar{\Gamma}^*_{11} X_{1t-1} + \bar{C}^*_1 + \bar{\Psi}^*_v \bar{V}_t + \bar{\Psi}^*_w \bar{W}_t, \\ E_t[Y_{t+1}] &= \bar{C}^*_2 + \bar{M}^* X_{1t}. \end{aligned} \tag{27}$$

To check observational equivalence of the true model and the equivalent model one must make sure that in any given example,

$$\begin{aligned} \Gamma^*_{11} &= \bar{\Gamma}^*_{11}, \quad C^*_1 = \bar{C}^*_1, \\ C^*_2 &= \bar{C}^*_2, \quad M^* = \bar{M}^*. \end{aligned}$$

The solution algorithm `FindEquiv` generates a matrix $\bar{\Omega}$ such that

$$\Psi^*_v I_l \Psi^{*\prime}_v = \left[\, \bar{\Psi}^*_v \; \bar{\Psi}^*_w \,\right] \bar{\Omega} \left[\, \bar{\Psi}^*_v \; \bar{\Psi}^*_w \,\right]'.$$

This equality implies that the reduced forms of the two systems are observationally equivalent when the DGP is driven by shocks V_t with covariance matrix I_l and the equivalent system is driven by shocks $\left[\bar{V}_t, \bar{W}_t\right]$ with covariance matrix $\bar{\Omega}$.

5. CONCLUSIONS

To summarize, this paper is about identification in linear rational expectations models. We provide an algorithm, implemented in Matlab, that generates equivalence classes of exactly identified models. This algorithm operates in three steps. First, the user specifies a "true" structural model, or data generating process. Second, the algorithm is used to calculate the parameters of a reduced form; these parameters are functions of the parameters of the structural model. Third, the user specifies an alternative economic theory in the form of a set of linear restrictions. The linear restrictions, in combination with the reduced form parameters, allow the user to generate an equivalent structural model that is observationally equivalent to the true DGP.

Observational equivalence is not a new concept in the rational expectations literature. However, we provide an example based on the new-Keynesian theory of the monetary transmission mechanism in which the true model and the equivalent model have different determinacy properties. In our example we establish an equivalence between a class of models proposed by Benhabib and Farmer

(2000) and the standard new-Keynesian model. This, we believe, is a new and disturbing result, because equilibria in the Benhabib–Farmer model are typically indeterminate for a class of policy rules that generate determinate outcomes in the new-Keynesian model.

NOTES

1. Examples include Clarida et al. (2000), Galí and Gertler (1999), and Fuhrer and Rudebusch (2004).

2. Examples of recent papers that make this, or related points, are those of Canova and Sala (2005), Lindé (2001, 2005), Lubik and Schorfheide (2004), Mavroedis (2002), and Nason and Smith (2003).

3. The monograph *Statistical Inference in Dynamic Economic Models* (1950), edited by Koopmans and Marschak, is an excellent collection that introduces many of the econometric ideas associated with the Cowles Commission.

4. This often-cited condition is usually sufficient to guarantee uniqueness. However, the necessary and sufficient conditions for existence and uniqueness are more complicated in general models and involve a spanning condition applied to a rotation of the model based on a QZ decomposition of the matrices A and B. For the exact conditions for existence and uniqueness the reader is referred to Sims (2002, pp. 11 and 12). The QZ decomposition for the square matrices A and B is a pair of upper triangular matrices S and T and a pair of orthonormal matrices Q and Z such that $QTZ = A$, $QSZ = B$ and $QQ' = ZZ' = I$. The ratios $[S_{ii}]/[T_{ii}]$ of the diagonal elements of S and T are referred to as generalized eigenvalues or roots.

5. Lubik and Schorfheide (2003) provide an extension of Sims's code `Gensys` that handles the case of indeterminacy.

6. The code used to generate the example in this section is available at http://farmer.sscnet.ucla.edu/NewWeb/Computer%20Code/WhatWeCodeMatlab/.

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APPENDIX

This section defines the matrices $H, h, J,$ and j from equations (22) and (24), Section 4:

$$\Bigg(\underset{l\times l}{[I_l]} \otimes \overset{H}{\underset{l(3l+1)\times l(5l+1)}{\underset{(3l+1)\times(5l+1)}{\left[\underset{((3l+1)\times 2l)}{\begin{matrix} \underset{2l\times 2l}{\Gamma^{*\prime}} \\ \underset{1\times 2l}{C^{*\prime}} \\ \underset{l\times 2l}{\Psi_v^{*\prime}} \end{matrix}} \;\vdots\; \underset{(3l+1)\times(3l+1)}{-I_{(3l+1)}}\right]}}}\Bigg) \underset{l(5l+1)\times 1}{\overset{\bar{\theta}}{\left(\operatorname{vec}\underset{(5l+1)\times l}{\left[\begin{matrix} \underset{l\times l}{A'} \\ \underset{l\times l}{F'} \\ \underset{l\times l}{B_1{}'} \\ \underset{l\times l}{B_2{}'} \\ \underset{1\times l}{C'} \\ \underset{l\times l}{\Psi_V{}'} \end{matrix}\right]}\right)}} = \overset{h}{\underset{l(3l+1)\times 1}{[0]}}, \tag{A.1}$$

$$\underset{l(5l+1)\times l(5l+1)}{\left(\begin{bmatrix} \underset{l(3l+1)\times l(5l+1)}{\begin{bmatrix}\underset{l\times l}{I_l}\end{bmatrix} \otimes \left(\underset{(3l+1)\times(5l+1)}{\begin{bmatrix} \underset{((3l+1)\times 2l)}{\begin{matrix} \underset{2l\times 2l}{\Gamma^{*\,\prime}} \\ \underset{1\times 2l}{C^{*\,\prime}} \\ \underset{l\times 2l}{\Psi_v^{*\prime}} \end{matrix}} & \vdots - I_{(3l+1)} \end{bmatrix}}\right)} \\ \underset{l(2l)\times l(5l+1)}{R} \end{bmatrix}\right)} \left(\operatorname{vec} \underset{(5l+1)\times l}{\begin{bmatrix} \underset{l\times l}{\bar{A}'} \\ \underset{l\times l}{\bar{F}'} \\ \underset{l\times l}{\bar{B}_1'} \\ \underset{l\times l}{\bar{B}_2'} \\ \underset{1\times l}{\bar{C}'} \\ \underset{l\times l}{\bar{\Psi}_v'} \end{bmatrix}}\right) = \underset{l(5l+1)\times 1}{\begin{pmatrix} \underset{l(3l+1)\times 1}{[0]} \\ \underset{l(2l)\times 1}{[r]} \end{pmatrix}}. \tag{A.2}$$