MONETARY AND FISCAL POLICY WHEN PEOPLE HAVE FINITE LIVES*

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Abstract. This paper incorporates realistic demographic structure into macroeconomic policy analysis by examining an overlapping generations model calibrated to match U.S. income data over the lifecycle. We provide explicit conditions for the existence of multiple dynamically efficient steady-state equilibria with positive debt levels for empirically relevant calibrations. Unlike representative agent frameworks, indeterminacy arises even with active monetary and fiscal policies. Following fundamental shocks, the model generates highly persistent swings in real interest rates consistent with evidence on long-horizon trends. Our tractable approach bridges theoretical models relying on infinite horizons and homogeneous agents with finite-lived heterogeneous consumers that populate actual economies.

1. Introduction

The theoretical models that underpin macroeconomic policy analysis typically consist of infinite-horizon rational agents that enable clean analytical results. However, a key question is whether insights gleaned from such models apply to real-world economies comprised of finite-lived agents with realistic demographic structures. This paper explores this issue by analyzing how monetary and fiscal policies interact with private sector choices in an overlapping generations (OLG) model calibrated to match U.S. demographic data.

We make several contributions. First, we prove theoretically in a simplified framework that the combination of a hump-shaped income profile and a sufficiently low intertemporal elasticity of substitution (IES) generates multiple balanced steady-state equilibria. Second, we construct a 62-period OLG model with capital accumulation where the income profile matches U.S. data and show for an empirically relevant IES that multiple indeterminate balanced steady states indeed arise. This violation of uniqueness occurs even with active monetary and fiscal policies, contrasting

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with standard deterministic representative agent models. Third, following shocks to fundamentals, the calibrated OLG model generates prolonged swings in real interest rates consistent with evidence.

To understand the practical relevance of our analysis, note that one keystone of modern macroeconomic theory is using comparative statics to predict how interventions shift economic outcomes. This exercise requires a unique mapping from fundamentals to equilibrium prices and quantities. However, we demonstrate that moving from common simplifying assumptions to finite lives and realistic demographics overturns basic determinacy properties that underlie policy evaluation. Consequently, caution should be exercised in applying conventional results. Our framework provides a foundation for constructing alternative macroeconomic models which can better account for actual demographic patterns.

The rest of the paper proceeds as follows. Section 2 discusses related literature and how our work departs from the seminal contribution of Aiyagari (1988). Section 3 explains the terminology around active and passive monetary and fiscal policies whilst Section 4 derives theoretical conditions for multiplicity in a simplified 3-period OLG model and explains the concept of indeterminacy.

Section 5 generalizes our framework to a T-period production economy and proves a theorem that allows us to compute the degree of indeterminacy. Section 6 calibrates a 62-period model to U.S. data and quantitatively demonstrates indeterminacy and prolonged real interest rate fluctuations even with active policy regimes. Importantly, our calibrated parameter values match estimates in the empirical literature. This section also checks the robustness of the key results across several alternative model calibrations. Section 7 incorporates monetary policy rules like the Taylor rule and derives additional restrictions on equilibria. Even with these additional restrictions we find that real indeterminacy persists. Section 8 concludes by discussing limitations of common arguments against relying on models with indeterminate or multiple equilibria, arguing that existing empirical evidence actually favors the parameter regions identified by our OLG model.

2. The Relationship of our Work to Previous Literature

The potential for multiple steady-state equilibria in overlapping generations (OLG) models has been long recognized since the seminal work of Samuelson (1958). And since the work of Kehoe and Levine (1985) we have known that these models can display relative price indeterminacy. However, most previous examples rely on highly stylized two or three period frameworks where indeterminacy is purely monetary and lacks quantitative relevance (Gale, 1973; Azariadis, 1981; Farmer, 1986).

This paper makes several key advances. First, we provide explicit sufficient theoretical conditions in a simplified three-period exchange economy for the emergence of multiple balanced steady states based on the hump shape of realistic income profiles over the lifecycle and a sufficiently low intertemporal elasticity of substitution (IES). Previous literature has lacked transparent analytical results for multiplicity linked directly to life-cycle income patterns.

Second, we prove that incorporating capital accumulation and production preserves indeterminacy and positive debt levels at steady states, even when both monetary and fiscal policies are active. This finding contrasts sharply with analogous deterministic representative agent models that rely on an active/passive policy mix for determinacy, highlighting the first-order role of demographic structure (Leeper, 1991; Woodford, 1994).

Third, we numerically verify the quantitative relevance of multiple indeterminate steady states in a realistically calibrated 62-period OLG model where incomes match U.S. data. To our knowledge, this is the first quantified demonstration in a reasonably sized OLG framework of the violations of uniqueness and determinacy that arise under standard calibrations. Our analysis showcases both the tractability and empirical plausibility of the mechanisms we identify.

Our results relate closely to several strands of literature. Chalk (2000) explores dynamic properties of a production-based 2-period OLG model. Under certain concavity assumptions on aggregate savings, he establishes generic existence of two steady states. We demonstrate explicitly that introducing an income profile consistent with evidence violates these conditions, generating further multiplicity.

Influential early work by Auerbach and Kotlikoff (1987) along with Ríos-Rull (1996) finds a unique calibrated steady state equilibrium in OLG models. They presume critical monotonicity properties of savings hold uniformly. We provide novel counterexamples displaying robust steady-state indeterminacy for empirically relevant IES values and debt levels, overturning conventional wisdom on the limited role of demographic realism (Aiyagari, 1988).

Kubler and Schmedders (2011) argue multiplicity generically disappears in economies with long but finite horizons if agents have randomly distributed endowments. However, we establish that retaining the hump-shaped pattern of incomes preserves indeterminacy despite expanding the cohort length towards infinity. Our quantified framework helps reconcile theory with evidence on the first-order relevance of demographic structure and finite lives (Reichlin, 1992).

Conceptually, our analysis also connects tightly to the Fiscal Theory of the Price Level (FTPL) literature initiated by Leeper (1991) and Woodford (1994). This literature relies fundamentally on conjectured determinacy emerging under active fiscal policy and passive monetary policy to pin down price levels. We provide a compelling counterexample demonstrating that their proposed determinacy arguments fail dramatically in realistic OLG economies once we account for demographic heterogeneity and finite planning horizons.

Overall, by incorporating crucial real world features of finite lives and age-varying income profiles motivated by microdata, our paper offers both novel theoretical results on the origins of multiplicity and indeterminacy as well as quantitative evidence of these forces in action. We provide a tractable foundation for building richer computational macroeconomic models complementing the dominant but demographically limited infinite horizon paradigm.

3. Fiscal and Monetary Policy

We begin by defining key fiscal and monetary policy variables that play a role in later characterizing fiscal-monetary policy regimes and how they interact with determinacy of equilibrium. Let B_t denote the stock of government debt in dollar terms and let τ_t be the level of lump-sum taxes.

The government budget constraint in real terms is then given by:

$$b_{t+1} = R_{t+1}(b_t + d_t), \quad t = 1, \ldots, \infty. \tag{1}$$

Here, $b_t \equiv B_{t-1}/p_t$ is the real bond value, $d_t = g_t - \tau_t$ is the real primary deficit where g_t denotes real spending, and R_{t+1} is the gross real interest rate. The evolution of government debt equals past debt that is rolled over plus the new issuance of debt to finance current deficits.

Monetary policy is summarized by a rule that sets the nominal interest rate i_t. We begin by studying the case where the central bank follows a constant interest rate peg, fixing it at a given level over time. This assumes a passive monetary policy regime. The resulting nominal interest rate then determines the evolution of the gross inflation rate, Π_{t+1}, through the Fisher equation, which links the gross real rate R_{t+1} and gross inflation by:

$$R_{t+1} \equiv \frac{1 + i_t}{\Pi_{t+1}}. \tag{2}$$

We can also write the government flow budget constraint as a single life-cycle constraint by consolidating the sequence of flow constraints and imposing the condition that the net present value of future surpluses is finite. Defining $\mathcal{Q}_t^{t+k}$ as the time t price for delivery of a commodity in period $t + k$, this leads to the following expression for the consolidated government budget constraint:

$$\frac{B_0}{p_1} = -\sum_{t=1}^{\infty} \mathcal{Q}_1^t d_t + \lim_{T \to \infty} \mathcal{Q}_1^T b_T. \tag{3}$$

If $\lim_{T \to \infty} \mathcal{Q}_1^T b_T$ exists and is equal to zero, Eqn. (3) defines a function relating the initial price level to the real value of all future surpluses.

In New-Keynesian models in which the central bank sets an interest rate peg, the initial price level would be indeterminate if the government were constrained to balance its budget for all paths of $\{\mathcal{Q}_1^t\}_{t=1}^{\infty}$ and all initial price levels (McCallum, 2001). To resolve this apparent indeterminacy of the price level, advocates of the FTPL have argued that Equation (3) should be treated not as a budget constraint, but as an equilibrium condition.[1] Although Equation (1) is expressed in terms of real variables, the debt instrument issued by the treasury is nominal. It follows that the real value of debt in period 1 is determined by the period 1 price level through the definition

$$b_1 \equiv \frac{B_0}{p_1}.$$

This argument rests critically on the determinacy properties of the steady state. In subsequent sections we will focus on the determinacy of equilibrium and on the interaction of fiscal and monetary policies under alternative assumptions about population demographics. In particular, we will provide novel counterexamples to established results about the determinacy of equilibria by incorporating finite lives and heterogeneity into an otherwise standard model of fiscal-monetary policy interactions.

[1] See, for example, (Leeper, 1991; Woodford, 1994, 2001). For a recent exposition of the FTPL see Cochrane (2023).

4. A Three-Period-Lived Overlapping Generations Model

We develop a 3-period overlapping generations model to illustrate existence of multiple steady-state equilibria. Preferences are given by the function

$$U = \frac{{c_t^t}^{1-\frac{1}{\eta}} + \beta {c_{t+1}^t}^{1-\frac{1}{\eta}} + \beta^2 {c_{t+2}^t}^{1-\frac{1}{\eta}}}{1-\frac{1}{\eta}}, \tag{4}$$

where β is the discount parameter and η is the IES. We index generations by superscripts and calendar time by subscripts. Thus, c_τ^t is the consumption of generation t in period τ.

Consumers maximize utility subject to three budget constraints, one for each period of life,

$$c_t^t + s_{t+1}^t \leq \omega_1, \quad c_{t+1}^t + s_{t+2}^t \leq R_{t+1} s_{t+1}^t + \omega_2, \quad c_{t+2}^t \leq R_{t+2} s_{t+2}^t + \omega_3, \tag{5}$$

where $\omega \equiv \{\omega_1, \omega_2, \omega_3\}$ is the after-tax endowment profile and s_τ^t is the demand for claims to $\tau + 1$ consumption goods by generation t in period τ. The subscript on the term ω_j indexes age and we assume throughout, that ω_j does not depend on calendar time. The solution to this problem is fully characterized by a pair of asset demand functions

$$s_{t+1}^t(R_{t+1}, R_{t+2}), \quad s_{t+2}^t(R_{t+1}, R_{t+2}),$$

together with the requirement that the three budget constraints characterized in (5) hold with equality.

Let the aggregate demand for assets by all agents alive at date t be defined by the function

$$S_\omega(R_t, R_{t+1}, R_{t+2}) \equiv s_t^{t-1}(R_t, R_{t+1}) + s_t^t(R_{t+1}, R_{t+2}),$$

where the subscript ω on the function S indexes the dependence of the asset demand function on the endowment profile. For this three-generation example, $S_\omega(\cdot)$ adds up the asset demand of the newborns, this is the term $s_t^t(\cdot)$, and the asset demands of the middle-aged, this is the term $s_t^{t-1}(\cdot)$. Equilibrium in the asset markets requires that

$$S_\omega(R_t, R_{t+1}, R_{t+2}) = b_t + d_t, \tag{6}$$

where $b_t + d_t$ is the public sector borrowing requirement in period t and the dynamics of public borrowing are given by the equation,

$$b_{t+1} = R_{t+1}(b_t + d_t). \tag{7}$$

Beginning with period 2, non-stationary equilibria are characterized by bounded sequences of real interest rates and debt that satisfy equations (6) and (7)and are consistent with a set of initial conditions that arise from the behavior of people alive in the initial period.

Let $f_\omega(R) \equiv S_\omega(R, R, R)$ be the aggregate *steady-state* demand for assets by the private sector. A steady-state equilibrium is a non-negative real number R and a real number b such that

$$f_\omega(R) = b + d, \quad \text{and} \quad b = R(b + d). \tag{8}$$

When $d = 0$, the second of the two equations (8) reduces to the expression $b = Rb$ which can generically be satisfied in one of two ways. Either $b = 0$ or $R = 1$.[2] We refer to an equilibrium in which $b = 0$ as a *balanced* steady state and the equilibrium in which $R = 1$ as the *golden rule* steady state.

We seek a sufficient condition for there to be multiple balanced steady-states. We choose an endowment pattern of $\omega = \{1, \lambda^{\eta}, \lambda^{2\eta}\}$, where $\lambda^{\eta} \in (0, 1)$ is the rate at which the endowment declines with age. By allowing the endowment to depend on the IES we simplify the expression for the interest rate in the balanced steady state. We show in Appendix A that the balanced steady-state interest rate for this economy is given by the expression $R_{bal} = \lambda/\beta$ and we denote the savings function at this endowment pattern as f_{ω_λ}.

Next, we look at alternative hump-shaped endowment profiles which we parameterize by $\tilde{\omega}_2$ and we construct an alternative economy with a balanced equilibrium of $R = R_{bal}$. To maintain the same balanced equilibrium as in the original economy we must ensure that the endowments add up to the same aggregate endowment and that the net present value of the wealth of a new-born individual is preserved. These two additional conditions allow us to solve for the first and third period endowments, $\tilde{\omega}_1$ and $\tilde{\omega}_3$, as functions of $\tilde{\omega}_2$. We denote the alternative endowment pattern by $\tilde{\omega} = \{\tilde{\omega}_1(\tilde{\omega}_2), \tilde{\omega}_2, \tilde{\omega}_3(\tilde{\omega}_2)\}$ and the savings function by $f_{\tilde{\omega}}$. The fact that both economies have the same balanced steady state implies that $f_{\omega_\lambda}(R_{bal}) = f_{\tilde{\omega}}(R_{bal}) = 0$.

We prove, in Appendix A, that for a low enough value of the IES, the slope of the aggregate savings function in our alternative economy changes sign as we move from a declining endowment to a hump shaped endowment profile. As this sign change occurs, two new balanced equilibria appear, one on each side of R_{bal}. This idea is illustrated in Figure 1. To construct this figure we used the special case of $\beta = \lambda = 1$. For this case the slope of the savings function at the steady state changes sign for values of

$$\eta < \frac{\tilde{\omega}_2 - 1}{4}. \tag{9}$$

In the top panel of Figure 1 we graph the function $f_{\tilde{\omega}}$ for $\eta = 1/3.5$, $\beta = 1$, $\lambda = 1$ and an endowment profile $\tilde{\omega} = \{0.335, 2.33, 0.335\}$. For these parameters, the necessary condition for the existence of multiple steady state equilibria, Inequality (9), is satisfied. In the bottom panel we plot the excess demand for goods for these parameter values. This panel illustrates that when $R = 1$, the golden rule steady state and the middle balanced steady state coincide. This property is reflected in the fact that the excess demand for goods is tangent to the zero line at $R = 1$. Varying β and or λ shifts both curves and leads to the separation of the golden rule steady-state equilibrium from the middle of the three balanced steady-state equilibria.

The fact that there exist multiple steady states does not imply anything about the determinacy properties of any one of them and for some parameter values there may exist other attracting sets

[2]The adjective 'generically' is required because there exists a set of measure zero in the parameter space for which the two steady states coincide. When $d \neq 0$, a continuity argument establishes that there is an open set $d \in (d_L, d_U)$, which contains $d = 0$, for which the number of steady-state equilibria and the determinacy properties of each steady-state equilibrium is the same as the case where $d = 0$. It follows that, as long as the primary budget deficit is not too large, our analysis of the properties of equilibria for the case of $d = 0$ carries over to the case where the treasury runs a primary deficit or a primary surplus.

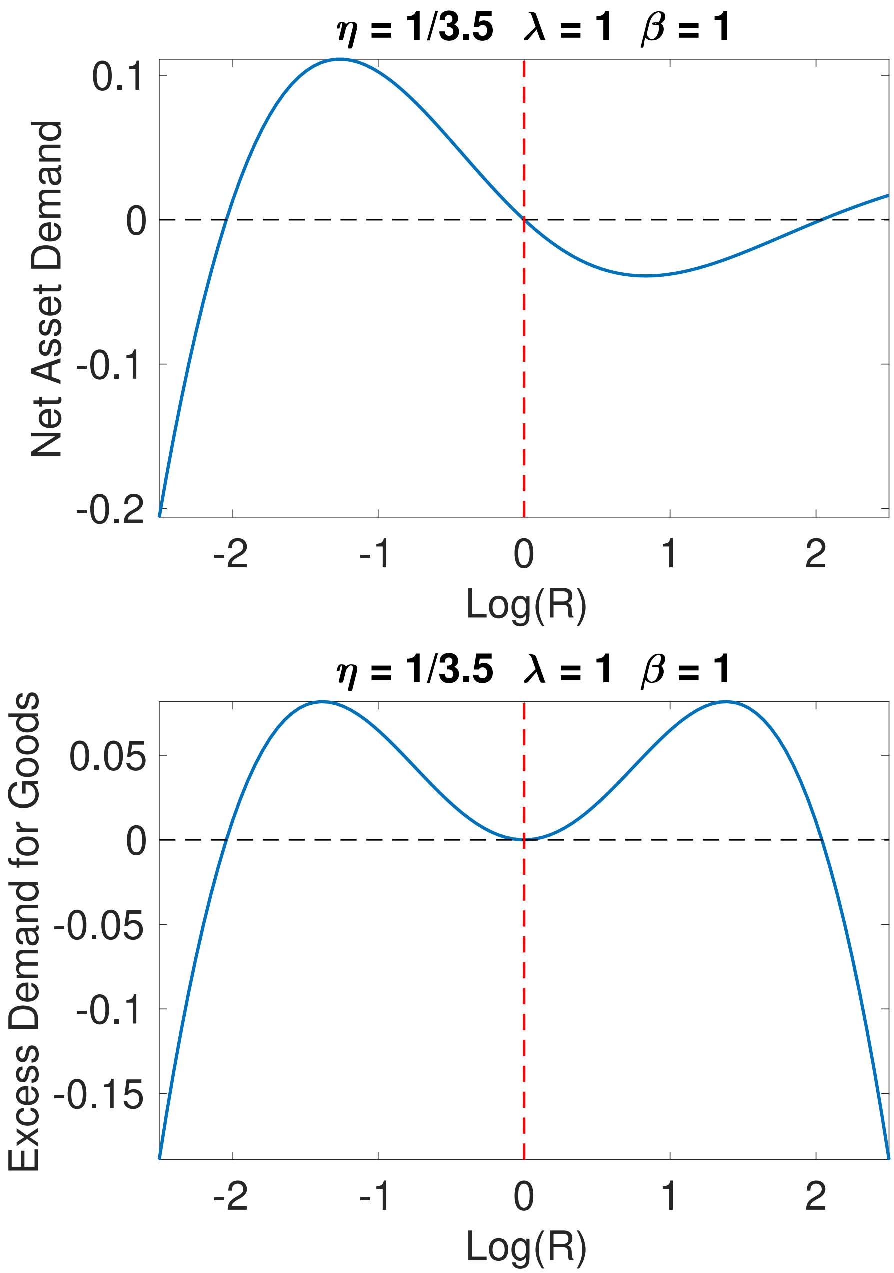

FIGURE 1. The Aggregate Savings Function for Parameter Values that Satisfy the Multiplicity Condition

including limit cycles and, possibly, chaotic attractors. However, we found computationally that, for a range of parameters, the golden rule steady state in our three-generation model displays second-degree indeterminacy for an active-passive policy combination. The following paragraphs define this concept and they explain its relevance for the theory of optimal fiscal and monetary policy.

Consider all pairs of initial values $b_1 \equiv B_0/p_1$ and R_2, that are close to the steady state values of b and R at the golden rule. If there is a unique pair, $\{b_1, R_2\}$, such that the trajectory that starts from this pair converges to the steady state, the golden rule steady state is said to be *locally determinate.* If there is a one dimensional manifold of values – defined by a function $b_1 = \phi(R_2)$ – such that all solutions to equations (6) and (7) that begin on this manifold converge to the steady state; the golden rule steady state is said to display one degree of indeterminacy. If there is a two dimensional manifold – containing the golden rule steady state – such that all solutions to equations (6) and (7) that begin on this manifold converge to the steady state; the steady state is said to display two degrees of indeterminacy.

The degree of indeterminacy of a steady state depends on the actions of the monetary and fiscal authorities. The assumption of a constant nominal interest rate implies that monetary policy is passive and the fact that d_t is not responsive to variations in the value of outstanding debt implies that fiscal policy is active. Arguably, this is the relevant policy mix for the recent policy environment in which the interest rate was at or near zero and unresponsive to realized inflation and where national treasuries were pursuing unrestrained spending programs that did not appear responsive to growing debt to GDP ratios. For this mix of an active fiscal and a passive monetary policy combination, our model displays *two-degrees of indeterminacy* at the golden rule steady state. In a representative agent economy, this policy mix would cause the steady state targeted by the monetary authorities to be determinate and it is that observation that was the initial impetus to the development of the FTPL.

In Section 7 we relax the assumption of a passive monetary policy and we show that the golden rule equilibrium still displays one degree of indeterminacy, even for the case in which monetary and fiscal policy are both active. This finding means that, although there is a unique equilibrium price sequence for every initial real interest rate, the real interest rate itself is not pinned down by fundamentals and it implies that the lessons of the FTPL are not robust to realistic changes in demographics.

5. The $T-$Period Production Economy

This section describes the generalization of the $3-$generation example to a production economy with T-generations. To handle this more general model we make two amendments to the exchange economy. First, we add a production sector and derive four functions that describe the dependence of the rental rate, the wage rate, the capital stock and output on the real interest rate. Second we explain how the addition of additional generations complicates the initial conditions. The main difference from our earlier $3-$generation example is that in the $T-$generation model, the initial conditions depend on the nominal asset positions of $T-2$ non-generic generations and on the initial price level.

We begin by describing the production sector. There is a unique commodity, denoted y_t, produced from labor, L_t and capital k_t by a large number of competitive firms using a Cobb-Douglas technology, $y_t = L_t^{1-\theta} k_t^{\theta}$, where θ is the elasticity of output with respect to capital. Labor is inelastically supplied and aggregate labor supply is fixed at $L_t = 1$. Let r_t be the real rental rate,

w_t the real wage rate and let δ represent the rate of capital depreciation. We show in Appendix C that profit maximization leads to the following four functions which describe the real rental rate, the real wage rate, the capital stock and output at date t as functions of R_t.

$$r_t = F_r(R_t) \equiv R_t - 1 + \delta, \qquad w_t = F_w(R_t) \equiv (1-\theta)\left(\frac{\theta}{F_r(R_t)}\right)^{\frac{\theta}{1-\theta}},$$
$$k_t = F_k(R_t) \equiv \left(\frac{\theta}{F_r(R_t)}\right)^{\frac{1}{1-\theta}}, \quad y_t = F_k(R_t)^{\theta}.$$

Next we turn to the household sector. Households are endowed with efficiency units of labor, distributed over the T periods of their lives according to the endowment profile $\{\omega_1, \omega_2, \ldots \omega_T\}$ where $\sum_{t=1}^{T} \omega_t = 1$. In our calibrated example we fix the weights ω_τ for $\tau = 1, \ldots, T$ to mirror the U.S. income distribution by age.

A generation t household solves the problem

$$\max_{\{c_t^t, \ldots, c_{t+T-1}^t\}} U^t(c_t^t, c_{t+1}^t, \ldots, c_{t+T-1}^t),$$

such that

$$c_t^t + s_{t+1}^t \leq \omega_1 F_w(R_t), \quad c_{t+1}^t + s_{t+2}^t \leq R_{t+1} s_{t+1}^t + \omega_2 F_w(R_{t+1}),$$
$$\ldots \quad c_{t+T-1}^t \leq R_{t+T-1} s_{t+T-1}^t + \omega_T F_w(R_{t+T-1}), \tag{10}$$

where $F_w(R_t)$ is the real wage at date t as a function of the gross interest rate. The solution to this problem is characterized by a set of $T-1$ savings functions, one for each of the first $T-1$ periods of life together with the requirement that the T budget constraints (10) hold with equality. In Appendix B, Section B.1, we characterize the solution to this problem for the case of CES preferences and we find an explicit formula for the aggregate asset demand function. This function, which we denote by $S(X_t)$ is the sum of the savings functions at date t for generations $t-T+2$ to t and it depends on the vector of interest rates $X_t \equiv \{R_{t-T+2}, \ldots, R_{t+T-1}\}$.

Define the private asset-demand function

$$F_A(X_t) \equiv S(R_{t-T+2}, \ldots R_{t+T-1}) - F_k(R_{t+1}). \tag{11}$$

A competitive equilibrium is a non-negative bounded sequence of real interest rates and a bounded sequence of net government bond demands that satisfies equations (12) and (13).

$$F_A(X_t) = b_t + d_t, \tag{12}$$
$$b_{t+1} = R_{t+1}(b_t + d_t). \tag{13}$$

Equation (12) characterizes sequences of real interest rates for which the net asset demand of the private sector is equal to the public sector borrowing requirement. Equation (13) describes the evolution of the public sector borrowing through time. These two equations differ from the representation of equilibrium in the 3−generation model in two ways. First, private savings may be held in the form of productive capital as well as in the form of government debt. This accounts for the appearance of the term $F_k(R_{t+1})$ in Equation (11). Second, the savings function of generation t depends on R_t through the dependence of the date t wage on the capital stock.[3]

In Appendix C.2 we show that dynamic equilibria can be described by a difference equation

$$F(X_t, X_{t-1}) \equiv F_A(X_t) - R_t F_A(X_{t-1}) + d_t = 0, \tag{14}$$

and we find a linear approximation to that difference equation around a steady state of the form

$$J_1 \tilde{X}_t = J_2 \tilde{X}_{t-1}, \tag{15}$$

where $\tilde{X}$ is a vector of deviations from a steady state and the matrices J_1 and J_2 are the Jacobians of $F(\cdot)$ with respect to X_t and X_{t-1} evaluated at this steady state.

The analysis in Appendix C.2 establishes that X_t has $2(T-1)$ elements and Appendix C.3 establishes that the initial conditions of the model place $T-1$ restrictions on the elements of X_1 and X_2. Using these results, in Appendix, D we prove the following proposition which is based on the work of Blanchard and Kahn (1980).

Proposition 1 (Blanchard-Kahn)**.** *Let K denote the number of generalized eigenvalues of* (J_1, J_2) *with modulus greater than* 1.[4]

- *If $K > T-1$ there are no bounded sequences that satisfy the equilibrium conditions in the neighbourhood of $\bar{X}$. In this case equilibrium does not exist.*
- *If $K = T-1$ there is a unique bounded sequence that satisfies the equilibrium equations. Further, this sequence converges asymptotically to the steady state $\left(\bar{R}, \bar{b}\right)$. In this case the steady state equilibrium $\left(\bar{R}, \bar{b}\right)$ is determinate.*
- *If $K \in \{0, \dots, T-2\}$ there is a $T-1-K$ dimensional subspace of initial conditions that satisfy the equilibrium equations. All of these initial conditions are associated with sequences that converge asymptotically to the steady state $\left(\bar{R}, \bar{b}\right)$. In this case the steady state equilibrium $\left(\bar{R}, \bar{b}\right)$ is indeterminate with degree of indeterminacy equal to $T-1-K$.*

It follows from Proposition 1 that we can compute the degrees of determinacy around a given steady-state equilibrium by comparing the number of generalized eigenvalues of (J_1, J_2) that lie outside the unit circle with $T-1$, where T is the number of generations. In the simulations presented in Section 6, we use this proposition to compute these generalized eigenvalues in the neighbourhood of each of the four steady states and we simulate non-stationary paths by iterating a linear approximation to the function $F(\cdot)$ around the golden-rule steady state.[5]

In our model, fiscal policy is active but monetary policy is passive. In a representative agent model, the FTPL dictates that this policy mix should lead to a unique initial price level. In Section 6 we provide an example of an OLG economy with a steady-state equilibrium where money has

[3]In the exchange economy, the functions $s_k^t(\cdot)$ for $k = t, \dots, t+T-2$ depend on $\{R_{t+1}, \dots, R_{t+T-1}\}$. In the model with production they depend on $\{R_t, \dots, R_{t+T-1}\}$. The extra term R_t appears because household income depends on w_t which is a function of R_t in equilibrium.

[4]The generalized eigenvalues of (J_1, J_2), are values of $\lambda \in \mathbb{C}$ that solve the equation $\det(J_1 - \lambda J_2) = 0$.

[5]The code used to generate all of our results is available online. Our code also replicates the findings reported in Kehoe and Levine (1983).

value and where the FTPL fails to hold. In this example, it is not only the initial price level that is indeterminate; it is also the initial real interest rate.

6. A Sixty-Two Generation Example

In this section we construct a sixty-two generation model where each generation begins its economic life at age 18 and in which a period corresponds to one year. We calibrate the age-profile of the representative person's endowment to U.S. data and we show that – for low values of the intertemporal elasticity of substitution – there exists a steady state that displays *real* indeterminacy, even when monetary and fiscal policy are both active.

We assume that the members of generation t maximize the utility function,

$$u\left(c_t^t, \ldots, c_{t+61}^t\right) = \sum_{i=1}^{62} \beta^{i-1} \left(\frac{{c_{t+i-1}^t}^{1-\frac{1}{\eta}} - 1}{1 - \frac{1}{\eta}} \right),$$

where η is the IES. The productivity of an agent's labor varies over the lifecycle and all labor is inelastically supplied to a competitive production sector which combines labor and capital in a Cobb Douglas technology. Households save by holding productive capital and government debt which are perfect substitutes. We calibrate the income profile of a representative generation to U.S. data and we provide explicit formulas for the excess demand functions for this functional form in Appendix B.

We graph our calibrated income profile in Figure 2. Our representative generation enters the labour force at age 18, retires at age 66, and lives to age 79. We chose the lifespan to correspond to current U.S. life expectancy at birth and we chose the retirement age to correspond to the age at which a U.S. adult becomes eligible for social security benefits. For the working-age portion of this profile we use data from Guvenen et al. (2021) which is available for ages 25 to 60. The working-age income profiles for ages 18 to 24 and for ages 61 to 66, were extrapolated to earlier and later years using log-linear interpolation. For the retirement portion we used data from the U.S. Social Security Administration.

U.S. retirement income comes from three sources; private pensions, government social security benefits, and Supplemental Security Income. We treat private pensions and government social security benefits as perfect substitutes for private savings since the amount received in retirement is a function of the amount contributed while working. To calibrate the available retirement income that is independent of contributions, we used Supplementary Security Income which, for the U.S., we estimate at 0.137% of GDP.[6]

For the remaining parameters of our model we chose a primary budget deficit of $d_t = 0$, an annual discount rate of 0.995 and an elasticity of substitution of 0.034. The qualitative features of the equilibria are robust to the existence of a positive primary deficit with an upper bound that

[6]From Table 2 of the March 2018 Social Security Administration Monthly Statistical Snapshot we learn that the average monthly Supplemental Security Income for recipients aged 65 or older equalled \$447 (with 2,240,000 claimants), which implies that total monthly nominal expenditure on Supplemental Security Income equalled \$1,003 million. This compares to seasonally adjusted wage and salary disbursements (A576RC1 from FRED) in February 2018 of \$8,618,700 million per annum, or \$718,225 million per month. Back of the envelope calculations suggest that Supplemental Security Income in retirement equalled 0.137% of total labour income.

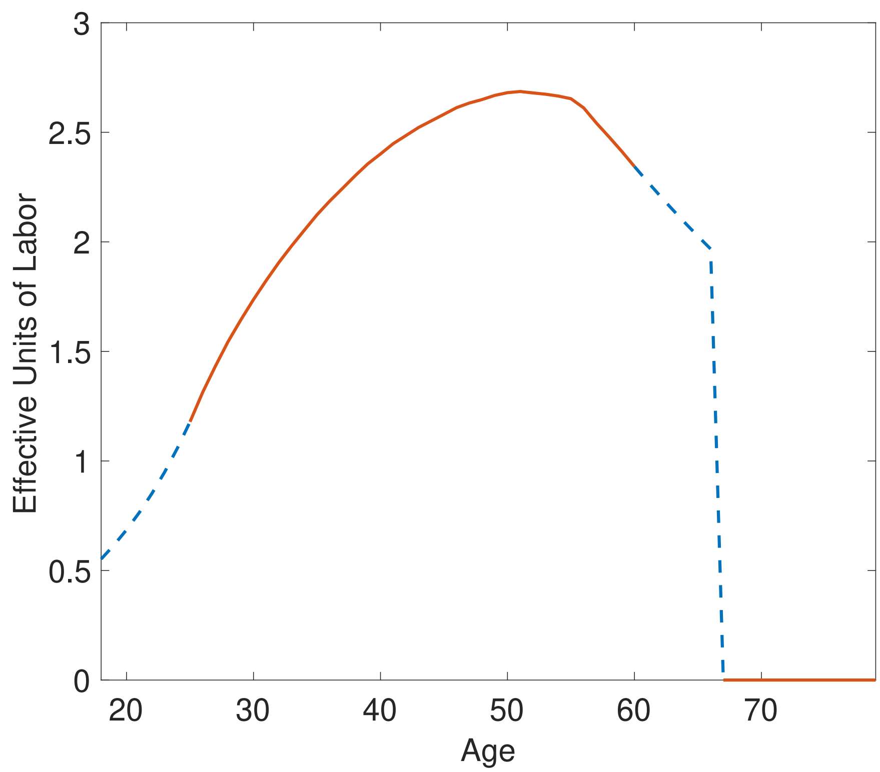

Figure 2. Normalized Endowment Profile. U.S. Data in Solid Red: Interpolated Data in Dashed Blue.

depends on the discount rate. For the calibrated income profile depicted in Figure 2 and for this choice of parameters, our model exhibits four steady-state equilibria. In Section 6 we explore the robustness of the properties of our model to alternative choices for the discount parameter and for the intertemporal elasticity of substitution.

The values and properties of all four steady-state equilibria are reported in Table 1. We refer to the balanced steady-state equilibria as Steady State A, Steady State C and Steady State D and to the golden-rule steady-state equilibrium as Steady State B. We see from this table that Steady States B, C and D are associated with a non-negative interest rate and are therefore dynamically efficient. Steady State A is associated with a negative interest rate of -9% and is therefore dynamically inefficient.[7]

The sixty-two generation production economy, with a calibrated income profile, is similar to the three generation endowment model from Section 4. In both models, the golden-rule steady-state equilibrium displays second degree indeterminacy. And in both models, the steady-state price level is positive and the initial price level is indeterminate even when fiscal policy is active. Importantly, because the monetary steady state is second-degree indeterminate, indeterminacy can hold even when both monetary and fiscal policy are active.

[7]See Cass (1972) for a definition and characterization of the conditions for dynamic efficiency.

Equilibrium Real Interest Rates					
Type	Interest Rate	Value of $\bar{b}$	# Unstable Roots	# Free Initial Conditions	Degree of Indeterminacy
Steady State A	-9%	0	60	61	1
Steady State B	0%	21 % of GDP	59	61	2
Steady State C	.004%	0	60	61	1
Steady State D	3.9%	0	61	61	0

Table 1. Steady States of the Sixty-Two Generation Model

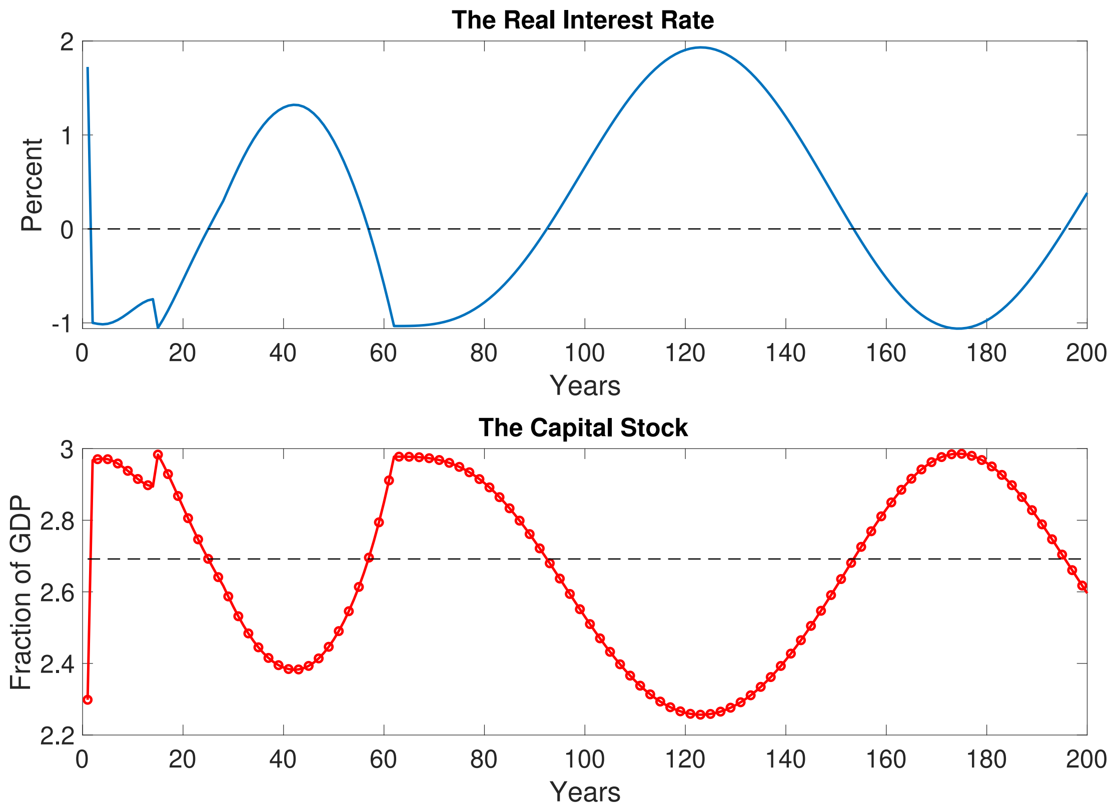

Figure 3. The Impact of a 1% Non-Fundamental Shock to the Initial Real Interest Rate

In Figure 3 we show the result of an experiment in which we perturb the initial value of the real interest rate by 1%, holding the price level, and the real wealth of all existing generations, fixed at their steady-state values. The upper panel of this Figure 3 plots the path by which the real interest rate returns to its steady-state value and the lower panel plots the return path of the value of physical capital expressed as a fraction of GDP. We refer to this perturbation as a 1% shock to the real interest rate.

This figure demonstrates that the return to the steady state following a relative price shock of this nature is extremely slow and that during the return the model displays prolonged periods of negative real interest rates.[8] This slow persistent return is generated by a pair of complex roots that are close to the unit circle and which *only* exist for calibrations of the model in which the steady-state equilibrium is indeterminate.

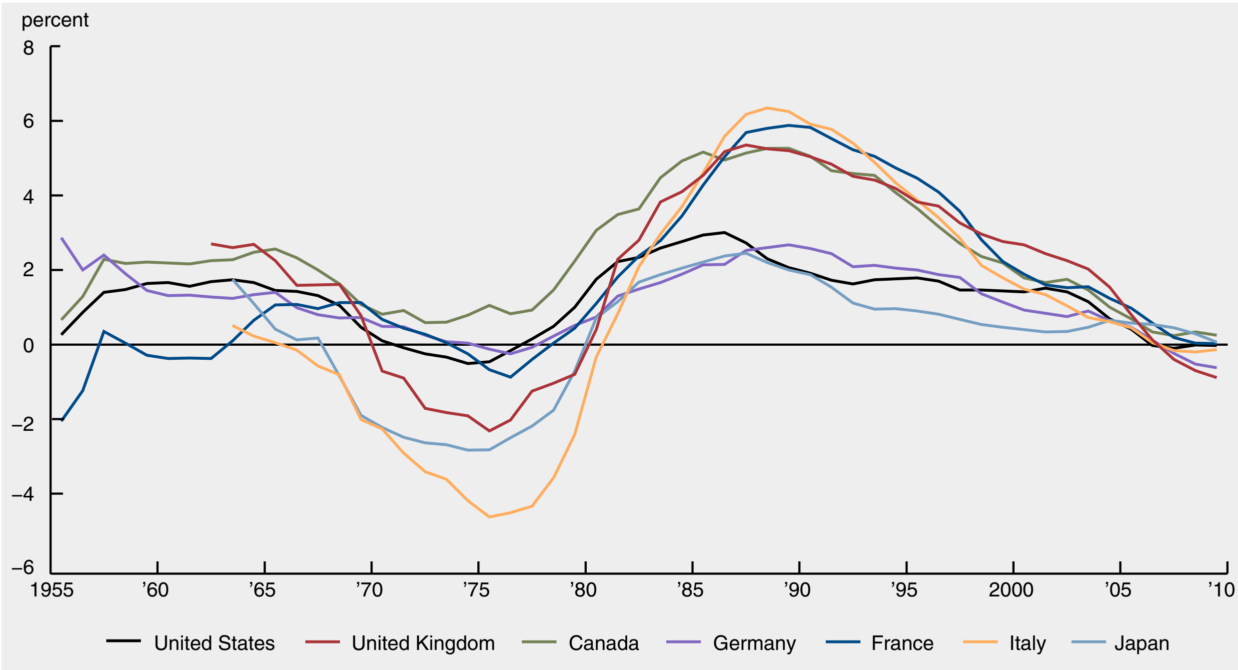

FIGURE 4. G7 Long-Run Real Interest Rates. Long-Run Real Interest Rates are 11-Year Centered Moving Averages of Annual Real Interest Rates.
Source: Figure 1 in Yi and Zhang (2017)

One may question whether the high degree of real interest rate persistence implied by our model is excessive. Have such long swings in real interest rates been observed in data? To address this question, Figure 4, reproduced from Yi and Zhang (2017), compares long run real interest rates in the G7 and documents that low-frequency real rate cycles, similar to those generated by our model, have characterized the evolution of real interest rates in all of these economies.[9]

To explore the robustness of our findings to alternative calibrations, in Table 2 we record the properties of our model for different values of the annual discount rate and the intertemporal elasticity of substitution. The example we featured in Section 6 had two degrees of indeterminacy and positive valued debt at the monetary steady state. Table 2 demonstrates that this property is not particularly special.

The table provides 40 different parameterizations of our model with intertemporal elasticity of substitution parameters ranging from .01 to .17 and discount rates ranging from 0.986 to 1. In all of these parameterizations we maintained the calibrated income profile illustrated in Figure

[8]Extending these simulations to much longer time periods confirms that these oscillations do eventually converge back to the steady state.

[9]See Yi and Zhang (2017) for a discussion of why long-run moving averages are likely to characterize trends in fundamental forces underlying real interest rates.

		Annual Discount Factor							
		0.986	0.988	0.990	0.992	0.994	0.996	0.998	1.00
IES									
	Degree of Indeterminacy	2	2	2	2	2	2	2	2
IES = 0.01	Value of Debt	18	19	19	20	21	21	22	23
	Degree of Indeterminacy	2	2	2	2	2	2	2	0
IES = 0.05	Value of Debt	1	4	7	10	13	16	20	23
	Degree of Indeterminacy	1	1	1	2	2	2	0	0
IES = 0.09	Value of Debt	-16	-10	-5	1	6	12	17	23
	Degree of Indeterminacy	1	1	1	1	1	2	0	0
IES = 0.13	Value of Debt	-33	-25	-7	-9	-1	7	15	23
	Degree of Indeterminacy	1	1	1	1	1	2	0	0
IES = 0.17	Value of Debt	-50	-39	-29	-19	-8	2	12	23

Table 2. Robustness of Indeterminacy to Alternative Calibrations at the Golden Rule Steady State

2. For each calibration Table 2 displays the number of degrees of indeterminacy and the value of government debt at the golden-rule steady-state equilibrium. There are thirteen parameterizations in which the golden-rule steady state displays one degree of indeterminacy and twenty in which it displays two degrees of indeterminacy. In all twenty of these parameterizations, debt has positive value in the steady state.

The calibrations in Table 2 demonstrate that our results require first, that the IES is low and second, that the discount factor is close to one. In our baseline calibration we choose IES = .034 and a value for the discount factor of 0.995. Both of these values are extreme by the standards of typical representative agent models but not outside the range of established estimates from the empirical literature. Thimme (2017) reviews a range of micro and macro estimates of the IES and he concludes that in “ ... almost every subsection of this paper we list studies that report estimates not significantly different from 0, as well as studies that report estimates above 1”.[10] Some of the studies cited by Thimme assume the existence of a representative agent, and others use micro-data sets. The main take away from his analysis is that there is no unique way to interpret the IES and that the value assigned to the parameter in an empirical study is context dependent.

A similar comment applies to our choice for the discount factor. In representative agent models there is a direct correspondence, between the rate of time preference and the rate of interest and a discount factor of 0.995 would be inconsistent with data. However, our model has an overlapping generations structure and in the OLG model there is no direct correspondence between these two objects. We conclude from this discussion that parameter values that make sense in calibrated representative agent models cannot be imported wholesale into the OLG environment. The parameters are measuring different things.

[10] Thimme (2017, pp 248).

7. Fiscal and Monetary Policy

In this section we discuss what happens when we relax either the assumption that fiscal policy is active or the assumption that monetary policy is passive. We first show that passive fiscal policy makes indeterminacy more likely. We then demonstrate that ensuring bounded inflation under an active Taylor rule imposes an additional restriction on the set of equilibrium paths. This additional restriction reduces the degree of indeterminacy by one.

Consider first what happens when fiscal policy is passive. To model a passive fiscal policy we assume that the treasury raises taxes, τ_t, in proportion to the real value of outstanding debt to ensure that the primary deficit d_t satisfies the equation

$$d_t = -\delta_b b_t,$$

where $\delta_b \geq 0$ is a debt repayment parameter. Combining this assumption with the government debt accumulation equation leads to the following amended debt accumulation equation,

$$b_{t+1} = [R_{t+1} - \delta_b] b_t.$$

For values of $[\bar{R} - \delta_b] < 1$ the effect of making fiscal policy passive is to introduce an additional stability mechanism that increases the degree of indeterminacy at each of the four steady states whenever δ_b is large enough. Passive fiscal policy makes indeterminacy more likely.

We next assume that fiscal policy is active and the central bank follows a Taylor rule (Taylor, 1999),

$$1 + i_t = \left(\frac{\bar{R}}{\bar{\Pi}^{\phi_\pi}} \right) \Pi_t^{1+\phi_\pi}, \quad t = 1, ...\infty. \tag{16}$$

Because this equation begins at date 1, the nominal interest rate in period 1 depends on p_0 through the definition, $\Pi_1 = p_1/p_0$. We treat p_0 as an initial condition that has the same status as the initial value of nominal debt, B_0. In Eq. (16), $\bar{\Pi}$ is the inflation target, $\bar{R}$ is the steady state real interest rate and ϕ_π is the response coefficient of the policy rate to deviations of inflation from target. The Taylor Rule is passive if $-1 \leq \phi_\pi \leq 0$ and active if $\phi_\pi > 0$.

When the central bank follows a Taylor Rule, the real interest rate and the real value of government debt continue to be determined by the bond market clearing equation and the debt accumulation equation. It follows that the conditions we have characterized in previous sections continue to ensure that the real interest rate and the real value of government debt remain bounded.

When the central bank follows a passive Taylor Rule, (see Appendix E.1) the following equation characterizes the asymptotic behaviour of the future inflation rate,

$$\lim_{T\to\infty} \tilde{\Pi}_{T+1} = \lim_{T\to\infty} (1 + \phi_\pi)^T \tilde{\Pi}_1 - \lim_{T\to\infty} \sum_{s=1}^{T} (1 + \phi_\pi)^{T-s} \tilde{R}_{T+1}, \tag{17}$$

where $\kappa \equiv \bar{\Pi}/\bar{R}$ and the tilde denotes deviations from the steady state. The limit of the first term on the right side of Equation (E2) is zero because $1 + \phi_\pi < 1$ and the second term is finite as a consequence of the boundedness of R_t. It follows that inflation is bounded whenever R_t is bounded. This is a generalization of the argument we made for the boundedness of the inflation rate when the central bank follows an interest rate peg and it does not impose any additional restrictions on the equations of the model for an equilibrium to be determinate.

When the central bank follows an active Taylor Rule, (see Appendix E.2), the initial price level is determined by the forward-looking equation

$$p_1 = p_0 \left(\bar{\Pi} + \kappa \sum_{s=1}^{+\infty} \left(\frac{1}{1+\phi_\pi} \right)^s \left(R_{1+s} - \bar{R} \right) \right). \tag{18}$$

Importantly, this restriction on the set of equilibrium paths is additional to the restriction

$$p_1 = \frac{B_0}{b_1},$$

that we used to generate the equilibrium sequence of interest rates. It follows that we are no longer free to pick R_2 and p_1 independently of each other. For any *given* choice of the initial interest rate, R_2, active monetary policy removes nominal indeterminacy. Crucially, however, it does not remove real indeterminacy and there continue to be many possible choices for the initial real interest rate, each of them associated with a different initial price level and a different equilibrium path for all future real interest rates and all future inflation rates.

8. Discussion and Implications

The findings that realistic demographic modeling overturns standard determinacy results and enables prolonged macroeconomic fluctuations after shocks raise several salient questions. We discuss the relevance of our analysis for theoretical macroeconomics and some potential broader implications.

For macroeconomic theory, a natural critique is that the real world evidently does not display the high degree of indeterminacy and price-level drift we characterize. This criticism presumes existing infinite horizon frameworks provide an accurate representation of the data and that the parameters can be accurately calibrated to match steady state ratios. However, we have shown that incorporating routine features of life-cycle demographics fundamentally alters model properties. Since applied modeling inherently abstracts from aspects of reality for tractability, the discipline may have been too quick to dismiss finite lives as largely unimportant. Our paper suggests this neglect risks missing essential drivers of macroeconomic behavior.

Relatedly, one could claim that determinacy should be used as model selection criteria, deeming indeterminate equilibria as less plausible. However, stability under learning dynamics does not uniquely favor determinacy Evans and Honkapohja (2001). And recent work shows convergence to Pareto optimal equilibria under deep reinforcement learning even when indeterminate Chen et al. (2021). Our analysis urges caution in applying such equilibrium selection arguments.

Furthermore, recent events like the prolonged period of extremely low global real rates suggest existing frameworks struggle to explain key prominent regularities. Our model endogenously generates the long-lasting swings in rates documented by Yi and Zhang (2017). More broadly, explicitly incorporating demographic factors like changing lifespans, retirement patterns, inequality etc. may prove indispensable for accurately tackling new empirical challenges in macroeconomics.

For policymakers, the fragility of some canonical determinacy and effectiveness results to realistic demography highlights risks in policy design relying solely on conventional models. Fiscal and monetary interventions could yield unintended consequences if actual economic responses differ from traditional predictions. Our paper provides a stepping stone towards improving policy guidance by capturing demographics. Expanding the dimensionality beyond representative agents to heterogeneous cohorts remains crucial future work as populations age.

In sum, revisiting long-held assumptions of infinite horizons and homogeneous agents seems imperative given empirical developments. We offer one plausible paradigm integrating lifecycle demographics. Substantial scope for research leveraging our approach remains, especially regarding normative analysis. But adequately reconciling theory and evidence appears difficult without finitely-lived models. Our findings urge rethinking traditional foundations underlying macroeconomic analysis.

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Appendix A. The Hump-shaped Profile and the IES

In this Appendix we establish that for the endowment profile $\omega_\lambda \equiv \{1, \lambda^\eta, \lambda^{2\eta}\}$ the balanced steady-state equilibrium occurs at $R = \lambda/\beta$. We find an alternative endowment profile with the same balanced steady-state and we find a sufficient condition for the alternative economy to have two additional balanced steady-state equilibria.

A.1. **Proof that $R = \lambda/\beta$ is a steady-state equilibrium for the endowment profile ω_λ.** We begin by deriving an expression for the steady-state savings function of an economy with a general endowment profile ω. Let

$$W_\omega(R) \equiv \frac{\omega_1}{1} + \frac{\omega_2}{R} + \frac{\omega_3}{R^2}, \tag{A1}$$

be the steady-state wealth of a new-born and define the function

$$\phi(R) = 1 + \frac{(\beta R)^\eta}{R} + \frac{(\beta R)^{2\eta}}{R^2}. \tag{A2}$$

Applying the solution to the T-generation maximizing problem with CES preferences from Appendix B we have the following steady-state consumption demand functions

$$c_1(R) = \psi(R), \quad c_2(R) = (\beta R)^\eta \psi(R), \quad c_3(R) = (\beta R)^{2\eta}\psi(R), \tag{A3}$$

$$\text{where} \quad \psi(R) \equiv \frac{W(R)}{\phi(R)}.$$

Define the steady-state savings functions of the young and middle-aged as

$$s_1(R) = \omega_1 - c_1(R), \quad s_2(R) = Rs_1(R) + \omega_2 - c_2(R). \tag{A4}$$

To establish that $R = \lambda/\beta$ is an equilibrium notice that $W_{\omega_\lambda}(\lambda/\beta) = \phi(\lambda/\beta) = 1 + \beta + \beta^2$. It follows from Eq. (A3) that $c_1 = 1$, $c_2 = \lambda^\eta$ and $c_3 = \lambda^{2\eta}$. This establishes that for a value of $R = \lambda/\beta$ each agent chooses to consume their endowment and thus $s_1(\lambda/\beta) = s_2(\lambda/\beta) = 0$ and $f_{\omega_\lambda}(\lambda/\beta) = 0$ characterizes a balanced steady-state equilibrium.

A.2. **Deriving an alternative endowment profile with the same balanced stead-state equilibrium.** We seek an alternative endowment pattern, $\tilde{\omega}$, which preserves $R = \lambda/\beta$ as a balanced steady-state.

It follows from Eqn. (A3) that consumption of each generation depends on $W_\omega(R)$, which is a function of the endowment pattern and $\phi(R)$, which is not. Hence if we choose an alternative endowment for which $W_{\tilde{\omega}}(R) = W_{\omega_\lambda}(R)$ the consumption demands of each generation will be the same as in the original economy when $R = \lambda/\beta$. This leads to one restriction on the choice of an alternative endowment pattern. A second restriction follows from the fact that, in equilibrium, the sum of the consumptions demands equals the aggregate endowment.

We parameterize the alternative endowment by $\tilde{\omega}_2$ and we seek functions $\tilde{\omega}_1(\tilde{\omega}_2)$ and $\tilde{\omega}_3(\tilde{\omega}_2)$ that satisfy these two restrictions. Eqn. (A5) restricts the wealth of a newborn to be the same in the two economies and Eqn. (A6) equates the aggregate endowments.

$$\tilde{\omega}_1 + \frac{\tilde{\omega}_2\beta}{\lambda} + \frac{\tilde{\omega}_3\beta^2}{\lambda^2}, = 1 + \frac{\lambda^\eta\beta}{\lambda} + \frac{\lambda^{2\eta}\beta^2}{\lambda^2}, \tag{A5}$$

$$\tilde{\omega}_1 + \tilde{\omega}_2 + \tilde{\omega}_3 = 1 + \lambda^\eta + \lambda^{2\eta}. \tag{A6}$$

The functions $\tilde{\omega}_1(\tilde{\omega}_2)$ and $\tilde{\omega}_3(\tilde{\omega}_2)$ that satisfy these two equations are given by the expressions,

$$\tilde{\omega}_1(\tilde{\omega}_2) = 1 + \frac{\frac{\beta}{\lambda}\lambda^\eta}{\left(1 + \frac{\beta}{\lambda}\right)} - \frac{\frac{\beta}{\lambda}\tilde{\omega}_2}{\left(1 + \frac{\beta}{\lambda}\right)}, \tag{A7}$$

$$\tilde{\omega}_3(\tilde{\omega}_2) = \frac{\lambda^\eta}{\left(1 + \frac{\beta}{\lambda}\right)} + \lambda^{2\eta} - \frac{\tilde{\omega}_2}{\left(1 + \frac{\beta}{\lambda}\right)}. \tag{A8}$$

A.3. **Evaluating the slope of $f_{\tilde{\omega}}(R)$ at the balanced steady-state equilibrium.** We seek a parametric restriction under which the slope of the function $f_{\tilde{\omega}}$ changes sign when evaluated at $R = \lambda/\beta$. To highlight the relationship between the IES and the peaked endowment profile we consider the parametric case when $\lambda = \beta = 1$. Aggregate savings is given by the expression,

$$f_{\tilde{\omega}}(R) = \Big(\tilde{\omega}_1(\tilde{\omega}_2) - \psi(R)\Big) + R\Big(\tilde{\omega}_1(\tilde{\omega}_2) - \psi(R)\Big) + \tilde{\omega}_2 - \psi(R)\Big(\beta R\Big)^\eta. \tag{A9}$$

Rearranging terms, this leads to the equation

$$f_{\tilde{\omega}}(R) = \tilde{\omega}_1(\tilde{\omega}_2)\Big(1 + R\Big) + \tilde{\omega}_2 - \psi(R)\Big(1 + R + (\beta R)^\eta\Big). \tag{A10}$$

For the parameter values $\lambda = \beta = 1$ the functions $\tilde{\omega}_1(\tilde{\omega}_2)$, $W_{\tilde{\omega}}(R)$ and $\phi(R)$ are given by the following formulae

$$\tilde{\omega}_1(\tilde{\omega}_2) = \frac{3 - \tilde{\omega}_2}{2}, \quad W_{\tilde{\omega}}(R) = 1 + \frac{1}{R} + \frac{1}{R^2}, \quad \phi(R) = 1 + R^{\eta-1} + R^{2(\eta-1)}. \tag{A11}$$

Evaluating each term at $R = \lambda/\beta = 1$ gives

$$W(1) = 3, \quad \phi(1) = 3, \quad \text{from which it follows that} \quad \psi(1) = 1. \tag{A12}$$

The partial derivatives of $W(R)$ and $\phi(R)$ are given by

$$\frac{\partial W}{\partial R} = -\frac{1}{R^2} - \frac{2}{R^3}, \quad \frac{\partial \phi}{\partial R} = (\eta - 1)R^{\eta-2} + 2(\eta - 1)R^{2\eta-3}, \tag{A13}$$

which when evaluated at $R = \lambda/\beta = 1$ gives

$$\left.\frac{\partial W}{\partial R}\right|_{R=1} = -3, \qquad \left.\frac{\partial \phi}{\partial R}\right|_{R=1} = 3(\eta - 1). \tag{A14}$$

Using the chain rule the partial derivative of $f_{\tilde{\omega}}(R)$ evaluated at the steady state $R = \lambda/\beta = 1$ is equal to

$$\left.\frac{\partial f_{\tilde{\omega}}}{\partial R}\right|_{R=1} = \tilde{\omega}_1(\tilde{\omega}_2) - \psi(1)\Bigg(1 + \eta\Bigg) - 3 \left.\frac{\partial \psi}{\partial R}\right|_{R=1}. \tag{A15}$$

A further application of the chain rule to the function $\psi(R)$ leads to the expression

$$\left.\frac{\partial \psi}{\partial R}\right|_{R=1} = \frac{\phi(1) \left.\frac{\partial W}{\partial R}\right|_{R=1} - W(1) \left.\frac{\partial \phi}{\partial R}\right|_{R=1}}{W(1)^2} = \frac{-9 - 9(\eta - 1)}{9} \quad = -\eta. \tag{A16}$$

Putting all these pieces together gives

$$\left.\frac{\partial f_{\tilde{\omega}}}{\partial R}\right|_{R=1} = \frac{3 - \tilde{\omega}_2}{2} - (1 + \eta) + 3\eta = \frac{1 - \tilde{\omega}_2 + 4\eta}{2}. \tag{A17}$$

Setting this expression less than zero leads to Inequality (9) in the body of the paper. The creation of two new steady states arises from the continuity of f_ω and the facts that $f_\omega(0) < 0$ and $f_\omega(R) > 0$ as $R \to \infty$. A continuous function that starts below zero, ends above zero and crosses zero from above at $R = \lambda/\beta$ must cross at least two more times, once for an interest rate less than λ/β and once for an interest rate great than λ/β.

□

Appendix B. Analytic Solutions for Excess Demand

B.1. **The generic optimization problem.** Consider a person with CES preferences who lives for T periods and has perfect foresight of future prices. This person solves the problem,

Problem 1.

$$\max_{\{c_t^t, c_{t+1}^t, \ldots, c_{t+T-1}^t\}} \frac{a_1(c_t^t)^\alpha + a_2(c_{t+1}^t)^\alpha + \ldots + a_T(c_{t+T-1}^t)^\alpha}{\alpha}, \tag{B1}$$

subject to the lifetime budget constraint

$$\sum_{i=1}^{T} \mathcal{Q}_t^{t-1+i} c_{t-1+i}^t = \sum_{i=1}^{T} \mathcal{Q}_t^{t-1+i} \omega_i F_w(R_i). \tag{B2}$$

Here, c_s^t is consumption in period s of a person born in period t, $i \in 1, \ldots T$ is age, and ω_i is the labor-endowment weight and $F_w(R_i)$ is the real wage at date i as a function of the gross real interest rate between periods $i-1$ and i. The parameters a_i are utility weights and $\alpha \leq 1$ is a curvature parameter which is related to intertemporal substitution, η, by the identity

$$\eta \equiv \frac{1}{1-\alpha}. \tag{B3}$$

The term $\mathcal{Q}_t^k$, defined by the expression

$$\mathcal{Q}_t^k \equiv \prod_{j=t+1}^{k} \frac{1}{R_j}, \qquad \mathcal{Q}_t^t = 1, \tag{B4}$$

is the relative price at date t of a commodity for delivery at date k.

This optimization problem includes the case of a constant discount factor β for which

$$[a_1, a_2 \ldots, a_T] = \left[1, \beta, \ldots, \beta^{T-1}\right] \tag{B5}$$

and logarithmic preferences which is the limiting case when $\alpha \to 0$. We permit the discount factor to vary with age to nest the Kehoe and Levine (1983) example which we use to cross-check our results.

Proposition 2. *The solution to Problem 1 is given by*

$$\hat{c}_{t-1+k}^t = \frac{a_k^\eta \sum_{i=1}^{T} \left(\mathcal{Q}_t^{t-1+i} \omega_i F_w(R_i)\right)}{\left(\mathcal{Q}_t^{t-1+k}\right)^\eta \sum_{i=1}^{T} \left(\mathcal{Q}_t^{t-1+i}\right)^{1-\eta} a_i^\eta}, \quad k = 1, \ldots, T. \tag{B6}$$

where $\hat{c}_{t-1+k}^t$ denotes the consumption, at time $t-1+k$, of an agent born at time t.

Proof. The result follows directly from substituting the first-order conditions into the budget constraint and rearranging terms. □

B.2. **Non-generic optimization problems.** Let j be an index that runs from 1 to $T-1$. Consider a non-generic person born in period $1-j$ with real assets $\frac{A_t^{1-j}}{p_1}$ who lives for $T-j$ periods. This person solves Problem 2

Problem 2.

$$\max_{\left\{c_1^{1-j},\ldots,c_{1-j+T-1}^{1-j}\right\}} \frac{a_{T-j+1}(c_1^{1-j})^\alpha + a_{T-j+2}(c_2^{1-j})^\alpha + \ldots + a_T(c_{1-j+T-1}^{1-j})^\alpha}{\alpha}, \quad j = 1\ldots, T-1 \tag{B7}$$

subject to the lifetime budget constraint

$$\sum_{k=1}^{(1-j)+T-1} \mathcal{Q}_1^k \left(c_k^{1-j} - \omega_{k-(1-j)+1} F_w(R_{k-(1-j)+1})\right) \le \frac{A^{1-j}}{p_1}. \tag{B8}$$

Proposition 3. *Let $k \in \{1,\ldots,T-j\}$. The solution to Problem 2 is given by*

$$\hat{c}_k^{1-j} = \frac{a_{k+j}^\eta \left(\frac{A_t^{1-j}}{p_1} + \sum_{i=1}^{T-j} \mathcal{Q}_t^i \omega_{j+i} F_w(R_{j+i})\right)}{\left(\mathcal{Q}_t^{t+k-1}\right)^\eta \sum_{i=1}^{T-j} \left(\mathcal{Q}_t^i\right)^{1-\eta} a_{j+i}^\eta}, \quad k = 1\ldots 1-j+T-1. \tag{B9}$$

Proof. The problem above is identical to a generic one solved by an agent who has $T-j$ periods to live, whose endowments are $\left\{\omega_{j+1}F_w(R_{j+1}) + \frac{A_t^{1-j}}{p_1}, \omega_{j+2}F_w(R_{j+2}), \ldots, \omega_T F_w(R_T)\right\}$, and whose preference parameters in the utility function are $\{a_{j+1}, a_{j+2}, \ldots, a_T\}$. □

Appendix C. Equilibrium as the Solution to a Difference Equation

In Section 4 we showed that equilibria of the 3−generation model can be characterized as the solution to a difference equation, determined by the behaviour of the generic generations, together with a set of initial conditions determined by the behavior of the non-generic generations. In this Appendix we generalize our analysis to the $T-$generation model with capital.

C.1. **Production.** Let output y_t be produced by the function

$$y_t = k_t^\theta L_t^{1-\theta}, \tag{C1}$$

and let capital depreciate at rate δ. Profit maximization leads to the expressions

$$w_t L_t = (1-\theta)y_t, \quad r_t k_t = \theta y_t, \tag{C2}$$

where r_t is the real rental rate and w_t is the real wage. No arbitrage implies

$$r_t = F_r(R_t) \equiv R_t - 1 + \delta, \tag{C3}$$

where R_t is the gross real rate of interest. By further rearranging equations (C1) – (C3) and imposing the labour supply equation, $L_t = 1$, we obtain the following expressions for w_t, k_t and y_t as functions of R_t;

$$w_t = F_w(R_t) \equiv (1-\theta)\left(\frac{\theta}{F_r(R_t)}\right)^{\frac{\theta}{1-\theta}}, \tag{C4}$$

$$k_t = F_k(R_t) \equiv \left(\frac{\theta}{F_r(R_t)}\right)^{\frac{1}{1-\theta}}, \tag{C5}$$

$$y_t = F_k(R_t)^{\theta}. \tag{C6}$$

C.2. **Equilibrium Difference Equation.** Define the vector X_t and the function $F_A(X_t)$,

$$X_t = [R_{t-T+2} \ldots, R_{t+T-1}]^{\top}, \tag{C7}$$

$$F_A(X_t) \equiv S(R_{t-T+2}, \ldots R_{t+T-1}) - F_k(R_{t+1}), \tag{C8}$$

where

$$S(R_{t-T+2}, \ldots, R_{t+T-1}) \equiv \sum_{\tau=t-T+2}^{t} s_t^{\tau}(R_{\tau}, R_{\tau+1}, \ldots, R_{\tau+T-1}), \tag{C9}$$

and the functional forms of the functions $s_t^{\tau}(\cdot)$ are derived by combining the solutions for the consumption functions from Appendix B with the fact that the sequence of budget constraints, (10), hold with equality.

Recall that a competitive equilibrium is characterized by a non-negative bounded sequence of real interest rates and a bounded sequence of net government bond demands that satisfies equations (C10) and (C11).

$$F_A(X_t) = b_t + d_t, \tag{C10}$$

$$b_{t+1} = R_{t+1}(b_t + d_t), \tag{C11}$$

and that a steady-state equilibrium is a non-negative real number $\bar{R}$ and a (possibly negative) real number $\bar{b}$ that solve the equations,

$$S(\bar{R}, \bar{R}, \ldots, \bar{R}) - F_k(R) = \bar{b} + d, \quad \bar{b}\,(1 - \bar{R}) = \bar{R}\,d. \tag{C12}$$

Let $\{\bar{R}, \bar{b}\}$ be a steady state equilibrium and let

$$\tilde{R}_t \equiv R_t - \bar{R}, \quad \text{and} \quad \tilde{b}_t \equiv b_t - \bar{b}, \tag{C13}$$

represent deviations of b_t and R_t from their steady state values. Define a function $F(\cdot)$,

$$F(X_t, X_{t-1}) \equiv F_A(X_t) - R_t F_A(X_{t-1}) + d_t, \tag{C14}$$

and let J_1 and J_2 represent the partial derivatives of this function with respect to X_t and X_{t-1}.

Using this notation, the local dynamics of equilibrium sequences close to the steady state can be approximated as solutions to the linear difference equation

$$J_1 \tilde{X}_t = J_2 \tilde{X}_{t-1}, \quad t = 2, \ldots \tag{C15}$$

with initial condition

$$\tilde{X}_1 = \bar{X}_1. \tag{C16}$$

The local stability of these equations depends on the spectrum of the matrix pencil (A, B), defined as solutions to the equation $\det(J_1 - \lambda J_2)$. We refer to the elements of the spectrum as generalized eigenvalues.

If one or more roots of $\lambda(J_1, J_2)$ are outside of the unit circle there is no guarantee that sequences of interest factors and government debt generated by Equation (C15) will remain bounded. To ensure stability, we must choose initial conditions that place $\tilde{X}_1$ in the linear subspace associated with the stable generalized eigenvalues of (J_1, J_2). The initial conditions are determined by the non-generic equilibrium conditions which we turn to next.

C.3. **Initial Conditions.** Asset market equilibrium in periods 1 through $T-1$ is characterized by a family of aggregate net savings functions, $G_{A_t}(\cdot)$, for $t = 1 \ldots T-1$ where $G_{A_t}(\cdot)$ is aggregate private savings net of the period $t+1$ capital stock. These functions are non-generic analogues of the function $F_A(X_t)$. They are different at each date because the asset demand functions of the initial generations depend on the initial wealth distribution and the initial price level as well as on real interest rates.

Consider the example of $T = 4$ which leads to the following asset market equilibrium equations in periods 1 through 4,

$$G_{A_1}\left(\frac{A^{-1}}{p_1}, \frac{A^0}{p_1}, k_1, R_2, R_3, R_4\right) - \frac{B_0}{p_1} - d_1 = 0, \tag{C18}$$

$$G_{A_2}\left(\frac{A^0}{p_1}, k_1, R_2, R_3, R_4, R_5\right) - R_2 G_{A_1}\left(\frac{A^{-1}}{p_1}, \frac{A^0}{p_1}, k_1, R_2, R_3, R_4\right) - d_2 = 0, \tag{C19}$$

$$G_{A_3}\left(k_1, R_2, R_3, R_4, R_5, R_6\right) - R_3 G_{A_2}\left(\frac{A^0}{p_1}, k_1, R_2, R_3, R_4, R_5\right) - d_3 = 0, \tag{C20}$$

$$F_A\left(R_2, R_3, R_4, R_5, R_6, R_7\right) - R_4 G_{A_3}\left(k_1, R_2, R_3, R_4, R_5, R_6\right) - d_4 = 0. \tag{C21}$$

The function G_{A_1} determines the net demand for government bonds in period 1 which must equal the net supply, $B_0/p_1 + d_1$. The period 1 capital stock enters the function G_{A_1} as a state variable that determines the date 1 real wage. The nominal liabilities A_1^{-1} and A^0 enter because generations -1 and 0 participate in the date 1 asset market. In period 2 the term A^{-1} is dropped from the function G_{A_2} because generation -1 does not enter the asset markets in their final period of life. The term R_5 enters this function because it enters the budget constraint of generation 2. In writing equations (C19) through (C21) we have used the government budget rule to substitute out for b_t, using the equality of debt with net asset demand from the previous period.

An equilibrium for the 4-generation production economy is a first order non-linear vector-valued difference equation in the 6 variables $X_t \equiv \{R_{t-2}, R_{t-1}, R_t, R_{t+1}, R_{t+2}, R_{t+3}\}$ with restrictions on X_1 and X_2 given by equations (C18) – (C21). These restrictions constitute a system of 4 equations in the 7 unknowns variables $p_1, R_2, R_3, R_4, R_5, R_6$ and R_7, leaving 3 free initial conditions.

Adding one period of life adds one additional period and one additional variable to this system of equations. The general result is that if people live for T periods, there are $T-1$ free initial conditions. An equilibrium for the T-generation production economy is characterized by a first-order vector-valued difference equation in the $2(T-1)$ variables $X_t \equiv \{R_{t-T+1}, \ldots, R_{t+T-1}\}$ with $T-1$ free initial conditions.

Appendix D. Proof of Proposition 1

Define the Jacobians $J_1(X_t, X_{t-1})$ and $J_2(X_t, X_{t-1})$

$$J_1(X_t, X_{t-1}) = \frac{\partial F(X_t, X_{t-1})}{\partial X_t}, \quad J_2(X_t, X_{t-1}) = \frac{\partial F(X_t, X_{t-1})}{\partial X_{t-1}}, \tag{D1}$$

and let

$$J_1^k = J_1(\bar{X}^k, \bar{X}^k), \quad \text{and} \quad J_2^k = J_2(\bar{X}^k, \bar{X}^k), \tag{D2}$$

be the values of the matrices J_1 and J_2 evaluated at steady state $\bar{X}^k$. In the following analysis, the dependence of J_1 and J_2 on k will be suppressed.

Using the generalized Schur decomposition, (Golub and Loan, 1996, page 377), define unitary matrices Q, and Z and upper triangular matrices S and T such that

$$J_1 = Q^\top S Z^\top, \quad \text{and} \quad J_2 = Q^\top T Z^\top. \tag{D3}$$

The spectrum $\lambda(J_1, J_2) \in \mathbb{C}$ is the set of solutions to the generalized eigenvalue problem $\det(J_1 - \lambda J_2) = 0$, and the values of λ are equal to the ratios of the diagonal elements of S and T.

Using Equation (D3) and the fact that $Q^\top Q$ and $Z^\top Z$ are identity matrices we can write the linear approximation to the function $F(\cdot)$ close to a steady state, Equation (15), as

$$SZ'\tilde{X}_t = TZ'\tilde{X}_{t-1}, \tag{D4}$$

which we break into stable and unstable blocks by ordering the Schur decomposition such that all of the elements of λ that are inside (outside) the unit circle appear in block 1 (block 2),

$$\begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{bmatrix} \begin{bmatrix} \tilde{Y}_t^1 \\ \tilde{Y}_t^2 \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix} \begin{bmatrix} \tilde{Y}_{t-1}^1 \\ \tilde{Y}_{t-1}^2, \end{bmatrix} \tag{D5}$$

where

$$\tilde{Y}_t^1 = Z^1 \tilde{X}_t, \quad \text{and} \quad \tilde{Y}_t^2 = Z^2 \tilde{X}_t, \tag{D6}$$

and the matrices Z^1 and Z^2 are a conformable partition of Z. Let K be the number of generalized eigenvalues with modulus greater than 1. To eliminate the effect of the unstable generalized eigenvalues on the dynamics of $\tilde{X}_t$ we set

$$\tilde{Y}_1^2 = 0, \tag{D7}$$

and notice that, from the block diagonality of S and T, if $\tilde{Y}_1^2 = 0$ then

$$\tilde{Y}_t^2 = (S_{22}{}^{-1}T_{22})^{t-1}\tilde{Y}_1^2 = 0 \quad \text{for all } t \geq 1. \tag{D8}$$

The requirement that equilibrium sequences remain bounded places K linear restrictions on the elements of $\tilde{X}$.

To compute the values of $\tilde{Y}_t^1$, we use Equation (D5) and the fact that $\tilde{Y}_t^2 = 0$ for all t to compute

$$\tilde{Y}_t^1 = (s_{11}^{-1}T_{11})^{t-1}\tilde{Y}_1^1. \tag{D9}$$

We established in Section C.3 that the non-generic equilibrium conditions place $T-1$ linear restrictions on the $2(T-1)$ elements of X_t leaving $T-1$ free initial conditions. It follows that the above construction is feasible and unique whenever $K = T-1$, infeasible if $K > T-1$ and that there are $T-1-K$ feasible choices for the initial conditions whenever $K < T-1$. This establishes Proposition 1.

Appendix E. Inflation Under a Taylor Rule

In this Appendix we derive equations that characterize the behaviour of the inflation rate when the Taylor Rule is passive and when it is active.

E.1. **The case of a passive Taylor Rule.** Using the Taylor rule to substitute for $1 + i_t$ in the Fisher parity condition yields the following difference equation for inflation

$$\Pi_{t+1} = \left(\frac{\bar{R}}{R_{t+1}}\right)\left(\frac{\Pi_t}{\bar{\Pi}}\right)^{\phi_\pi}\Pi_t, \quad \text{for all} \quad t = 1, \ldots \infty \tag{E1}$$

which we linearize around a steady state to obtain

$$\tilde{\Pi}_{t+1} = (1 + \phi_\pi)\,\tilde{\Pi}_t - \kappa\tilde{R}_{t+1}, \quad \text{for all} \quad t = 1, \ldots, \infty. \tag{E2}$$

Here, $\kappa \equiv \bar{\Pi}/\bar{R}$ and the tilde denotes deviations from the steady state. Iterating Equation (E2) we obtain

$$\lim_{T\to\infty} \tilde{\Pi}_{T+1} = \lim_{T\to\infty} (1+\phi_\pi)^T \tilde{\Pi}_1 - \lim_{T\to\infty} \sum_{s=1}^{T} (1+\phi_\pi)^{T-s} \tilde{R}_{T+1}. \tag{E3}$$

This is Equation (17) in Section 7.

E.2. **The case of an active Taylor Rule.** To find conditions under which inflation is bounded when the Taylor Rule is active, we use Equation E2 to write the inflation rate at date t as a function of all future real interest rates and all future inflation rates,

$$\tilde{\Pi}_t = \kappa \sum_{s=1}^{+\infty} \left(\frac{1}{1+\phi_\pi}\right)^s \tilde{R}_{t+s} + \lim_{T\longrightarrow\infty} \left(\frac{1}{1+\phi_\pi}\right)^T \tilde{\Pi}_{t+T}. \tag{E4}$$

If inflation is bounded, and if the Taylor Rule is active, the second term on the right side of Equation (18) is zero. Evaluating Equation (E4) at $t = 1$, we arrive the following expression for the initial gross inflation rate.

$$\tilde{\Pi}_1 \equiv \left(\tilde{\Pi}_1 - \bar{\Pi}\right) = \kappa \sum_{s=1}^{+\infty} \left(\frac{1}{1+\phi_\pi}\right)^s \tilde{R}_{1+s} \tag{E5}$$

Using the definition of inflation in period 1, Equation (E5) places the following restriction on the initial price level,

$$p_1 = p_0 \left(\bar{\Pi} + \kappa \sum_{s=1}^{+\infty} \left(\frac{1}{1+\phi_\pi}\right)^s \left(R_{1+s} - \bar{R}\right)\right). \tag{E6}$$

This is Equation (18) in Section 7.

□